

# Integrals

## Question1

If  $\int_0^1 \frac{1}{\sqrt{3+x} + \sqrt{1+x}} dx = a + b\sqrt{2} + c\sqrt{3}$ , where a, b, c are rational numbers, then  $2a + 3b - 4c$  is equal to :

[27-Jan-2024 Shift 1]

Options:

A.

4

B.

10

C.

7

D.

8

Answer: D

Solution:

$$\int_0^1 \frac{1}{\sqrt{3+x} + \sqrt{1+x}} dx = \int_0^1 \frac{\sqrt{3+x} - \sqrt{1+x}}{(3+x) - (1+x)} dx$$

$$\frac{1}{2} \left[ \int_0^1 \sqrt{3+x} dx - \int_0^1 (\sqrt{1+x}) dx \right]$$

$$\frac{1}{2} \left[ 2 \frac{(3+x)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2(1+x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1$$

$$\frac{1}{2} \left[ \frac{2}{3}(8 - 3\sqrt{3}) - \frac{2}{3}(2^{\frac{3}{2}} - 1) \right]$$

$$\frac{1}{3}[8 - 3\sqrt{3} - 2\sqrt{2} + 1]$$

$$= 3 - \sqrt{3} - \frac{2}{3}\sqrt{2} = a + b\sqrt{2} + c\sqrt{3}$$

$$a = 3, b = -\frac{2}{3}, c = -1$$

$$2a + 3b - 4c = 6 - 2 + 4 = 8$$

## Question2



If  $(a, b)$  be the orthocentre of the triangle whose vertices are  $(1, 2)$ ,  $(2, 3)$  and  $(3, 1)$ , and

$I_1 = \int_a^b x \sin(4x - x^2) dx$ ,  $I_2 = \int_a^b \sin(4x - x^2) dx$ , then  $36 \frac{I_1}{I_2}$  is equal to :

[27-Jan-2024 Shift 1]

Options:

- A.  
72
- B.  
88
- C.  
80
- D.  
66

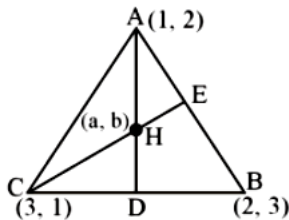
Answer: A

Solution:

Equation of CE

$$y - 1 = -(x - 3)$$

$$x + y = 4$$



orthocentre lies on the line  $x + y = 4$

so,  $a + b = 4$

$$I_1 = \int_a^b x \sin(x(4-x)) dx \dots\dots\dots(i)$$

Using king rule

$$I_1 = \int_a^b (4-x) \sin(x(4-x)) dx \dots\dots\dots(ii)$$

(i) + (ii)

$$2I_1 = \int_a^b 4 \sin(x(4-x)) dx$$

$$2I_1 = 4I_2$$

$$I_1 = 2I_2$$

$$\frac{I_1}{I_2} = 2$$

$$\frac{36I_1}{I_2} = 72$$

### Question3

For  $0 < a < 1$ , the value of the integral  $\int_0^{\pi} \frac{dx}{1 - 2a \cos x + a^2}$  is :

[27-Jan-2024 Shift 2]

Options:

A.

$$\frac{\pi^2}{\pi + a^2}$$

B.

$$\frac{\pi^2}{\pi - a^2}$$

C.

$$\frac{\pi}{1 - a^2}$$

D.

$$\frac{\pi}{1 + a^2}$$

Answer: C

Solution:

$$I = \int_0^{\pi} \frac{dx}{1 - 2a \cos x + a^2}; 0 < a < 1$$

$$I = \int_0^{\pi} \frac{dx}{1 + 2a \cos x + a^2}$$

$$2I = 2 \int_0^{\pi/2} \frac{2(1+a^2)}{(1+a^2)^2 - 4a^2 \cos^2 x} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{2(1+a^2) \cdot \sec^2 x}{(1+a^2)^2 \cdot \sec^2 x - 4a^2} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{2 \cdot (1+a^2) \cdot \sec^2 x}{(1+a^2)^2 \cdot \tan^2 x + (1-a^2)^2} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\frac{2 \cdot \sec^2 x}{1+a^2} \cdot dx}{\tan^2 x + \left(\frac{1-a^2}{1+a^2}\right)^2}$$

$$\Rightarrow I = \frac{2}{(1-a^2)} \left[ \frac{\pi}{2} - 0 \right]$$

$$I = \frac{\pi}{1-a^2}$$



## Question4

The integral  $\int \frac{(x^8 - x^2) dx}{(x^{12} + 3x^6 + 1)\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)}$  is equal to :

[27-Jan-2024 Shift 2]

Options:

A.

$$\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right)^{1/3} + C$$

B.

$$\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right)^{1/2} + C$$

C.

$$\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right) + C$$

D.

$$\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right)^3 + C$$

Answer: A

Solution:

$$I = \int \frac{x^8 - x^2}{(x^{12} + 3x^6 + 1)\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)} dx$$

$$\text{Let } \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) = t$$

$$\Rightarrow \frac{1}{1 + \left(x^3 + \frac{1}{x^3}\right)^2} \cdot \left(3x^2 - \frac{3}{x^4}\right) dx = dt$$

$$\Rightarrow \frac{x^6}{x^{12} + 3x^6 + 1} \cdot \frac{3x^6 - 3}{x^4} dx = dt$$

$$I = \frac{1}{3} \int \frac{dt}{t} = \frac{1}{3} \ln|t| + C$$

$$I = \frac{1}{3} \ln\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right| + C$$

$$I = \ln\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|^{1/3} + C$$

Hence option (1) is correct

## Question5

For  $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,

$$\text{if } y(x) = \int \frac{\operatorname{cosec} x + \sin x}{\operatorname{cosec} x \sec x + \tan x \sin^2 x} dx$$

and  $\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} y(x) = 0$  then  $y\left(\frac{\pi}{4}\right)$  is equal to

[29-Jan-2024 Shift 1]

Options:

A.

$$\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

B.

$$\frac{1}{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

C.

$$-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

D.

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(-\frac{1}{2}\right)$$

Answer: D

Solution:

$$y(x) = \int \frac{(1 + \sin^2 x) \cos x}{1 + \sin^4 x} dx$$

Put  $\sin x = t$

$$= \int \frac{1+t^2}{t^4+1} dt = \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{t-\frac{1}{t}}{\sqrt{2}} \right) + C$$

$$x = \frac{\pi}{2}, t = 1 \therefore C = 0$$

$$y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \tan^{-1} \left(-\frac{1}{2}\right)$$

---

## Question6

$$\text{If } \int \frac{\sin^{\frac{3}{2}}x + \cos^{\frac{3}{2}}x}{\sqrt{\sin^3x \cos^3x \sin(x-\theta)}} dx = A\sqrt{\cos\theta \tan x - \sin\theta} + B\sqrt{\cos\theta - \sin\theta \cot x} + C,$$

where  $C$  is the integration constant, then  $AB$  is equal to

[29-Jan-2024 Shift 2]

Options:

A.

4 cosec(2θ)

B.

4 secθ

C.

2 secθ

D.

8 cosec(2θ)

Answer: D

Solution:

$$\int \frac{\sin^{\frac{3}{2}}x + \cos^{\frac{3}{2}}x}{\sqrt{\sin^3x \cos^3x \sin(x-\theta)}} dx$$

$$I = \int \frac{\sin^{\frac{3}{2}}x + \cos^{\frac{3}{2}}x}{\sqrt{\sin^3x \cos^3x (\sin x \cos\theta - \cos x \sin\theta)}} dx$$

$$= \int \frac{\sin^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x \cos^2x \sqrt{\tan x \cos\theta - \sin\theta}} dx + \int \frac{\cos^{\frac{3}{2}}x}{\sin^2x \cos^{\frac{3}{2}}x \sqrt{\cos\theta - \cot x \sin\theta}} dx$$

$$= \int \frac{\sec^2x}{\sqrt{\tan x \cos\theta - \sin\theta}} dx + \int \frac{\operatorname{cosec}^2x}{2} \sqrt{\cos\theta - \cot^2x \sin\theta} dx$$

$I = I_1 + I_2$  ..... { Let }

For  $I_1$ , let  $\tan x \cos\theta - \sin\theta = t^2$

$$\sec^2x dx = \frac{2t dt}{\cos\theta}$$

For  $I_2$ , let  $\cos\theta - \cot x \sin\theta = z^2$

$$\operatorname{cosec}^2x dx = \frac{2z dz}{\sin\theta}$$

$$I = I_1 + I_2$$

$$= \int \frac{2t dt}{\cos\theta} + \int \frac{2z dz}{\sin\theta}$$

$$= \frac{2t}{\cos\theta} + \frac{2z}{\sin\theta}$$

$$= 2 \sec\theta \sqrt{\tan x \cos\theta - \sin\theta} + 2 \operatorname{cosec}\theta \sqrt{\cos\theta - \cot x \sin\theta}$$

Comparing

$$AB = 8 \operatorname{cosec} 2\theta$$


---

## Question 7

Let  $f(x) = \int_0^x g(t) \log_e \left( \frac{1-t}{1+t} \right) dt$ , where  $g$  is a continuous odd function.

If  $\int_{-\pi/2}^{\pi/2} \left( f(x) + \frac{x^2 \cos x}{1+e^x} \right) dx = \left( \frac{\pi}{\alpha} \right)^2 - \alpha$ , then  $\alpha$  is equal to.....

**[27-Jan-2024 Shift 2]**

**Options:**

**Answer: 2**

**Solution:**

$$f(x) = \int_0^x g(t) \ln \left( \frac{1-t}{1+t} \right) dt$$

$$f(-x) = \int_0^{-x} g(t) \ln \left( \frac{1-t}{1+t} \right) dt$$

$$f(-x) = -\int_0^x g(-y) \ln \left( \frac{1+y}{1-y} \right) dy$$

$$= -\int_0^x g(y) \ln \left( \frac{1-y}{1+y} \right) dy \quad (g \text{ is odd})$$

$$f(-x) = -f(x) \Rightarrow f \text{ is also odd}$$

Now,

$$I = \int_{-\pi/2}^{\pi/2} \left( f(x) + \frac{x^2 \cos x}{1+e^x} \right) dx \dots\dots(1)$$

$$I = \int_{-\pi/2}^{\pi/2} \left( f(-x) + \frac{x^2 e^x \cos x}{1+e^x} \right) dx \dots\dots(2)$$

$$2I = \int_{-\pi/2}^{\pi/2} x^2 \cos x dx = 2 \int_0^{\pi/2} x^2 \cos x dx$$

$$I = (x^2 \sin x)_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x dx$$

$$= \frac{\pi^2}{4} - 2(-x \cos x + \int \cos x dx)_0^{\pi/2}$$

$$= \frac{\pi^2}{4} - 2(0+1) = \frac{\pi^2}{4} - 2 \Rightarrow \left( \frac{\pi}{2} \right)^2 - 2$$

$$\therefore \alpha = 2$$

## Question 8

If the value of the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{x^2 \cos x}{1+\pi^x} + \frac{1 + \sin^2 x}{1 + e^{\sin x^{2023}}} \right) dx = \frac{\pi}{4}(\pi + a) - 2,$$

then the value of  $a$  is

**[29-Jan-2024 Shift 1]**

**Options:**

A.

3

B.

$$-\frac{3}{2}$$

C.

2

D.

3/2

**Answer: A**

**Solution:**

$$I = \int_{-\pi/2}^{\pi/2} \left( \frac{x^2 \cos x}{1 + \pi^x} + \frac{1 + \sin^2 x}{1 + e^{\sin x^{2023}}} \right) dx$$

$$I = \int_{-\pi/2}^{\pi/2} \left( \frac{x^2 \cos x}{1 + \pi^{-x}} + \frac{1 + \sin^2 x}{1 + e^{\sin(-x)^{2023}}} \right) dx$$

On Adding, we get

$$2I = \int_{-\pi/2}^{\pi/2} (x^2 \cos x + 1 + \sin^2 x) dx$$

On solving

$$I = \frac{\pi^2}{4} + \frac{3\pi}{4} - 2$$

a = 3

## Question9

If  $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1 - \sin 2x} dx = \alpha + \beta\sqrt{2} + \gamma\sqrt{3}$ , where  $\alpha, \beta$  and  $\gamma$  are rational numbers, then  $3\alpha + 4\beta - \gamma$  is equal to \_\_\_\_

[29-Jan-2024 Shift 2]

**Answer: 6**

**Solution:**



$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1 - \sin 2x} \, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} |\sin x - \cos x| \, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cos x - \sin x) \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sin x - \cos x) \, dx$$

$$= -1 + 2\sqrt{2} - \sqrt{3}$$

$$= \alpha + \beta\sqrt{2} + \gamma\sqrt{3}$$

$$\alpha = -1, \beta = 2, \gamma = -1$$

$$3\alpha + 4\beta - \gamma = 6$$


---

## Question 10

The value of  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n^3}{(n^2 + k^2)(n^2 + 3k^2)}$  is :

[30-Jan-2024 Shift 1]

Options:

A.

$$\frac{(2\sqrt{3} + 3)\pi}{24}$$

B.

$$\frac{13\pi}{8(4\sqrt{3} + 3)}$$

C.

$$\frac{13(2\sqrt{3} - 3)\pi}{8}$$

D.

$$\frac{\pi}{8(2\sqrt{3} + 3)}$$

Answer: B

Solution:

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n^3}{n^4 \left(1 + \frac{k^2}{n^2}\right) \left(1 + \frac{3k^2}{n^2}\right)} \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{n^3}{\left(1 + \frac{k^2}{n^2}\right) \left(1 + \frac{3k^2}{n^2}\right)} \\
&= \int_0^1 \frac{dx}{3(1+x^2) \left(\frac{1}{3} + x^2\right)} \\
&= \int_0^1 \frac{1}{3} \times \frac{3}{2} \frac{(x^2+1) - \left(x^2 + \frac{1}{3}\right)}{(1+x^2) \left(x^2 + \frac{1}{3}\right)} dx \\
&= \frac{1}{2} \int_0^1 \left[ \frac{1}{x^2 + \left(\frac{1}{\sqrt{3}}\right)^2} - \frac{1}{1+x^2} \right] dx \\
&= \frac{1}{2} [\sqrt{3} \tan^{-1}(\sqrt{3}x)]_0^1 - \frac{1}{2} (\tan^{-1}x)_0^1 \\
&= \frac{\sqrt{3}}{2} \left(\frac{\pi}{3}\right) - \frac{1}{2} \left(\frac{\pi}{4}\right) = \frac{\pi}{2\sqrt{3}} - \frac{\pi}{8} \\
&= \frac{13\pi}{8 \cdot (4\sqrt{3} + 3)}
\end{aligned}$$


---

## Question 11

The value  $9 \int_0^9 \left[ \sqrt{\frac{10x}{x+1}} \right] dx$ , where  $[t]$  denotes the greatest integer less than or equal to  $t$ , is

[30-Jan-2024 Shift 1]

Answer: 155

Solution:

$$\frac{10x}{x+1} = 1 \Rightarrow x = \frac{1}{9}$$

$$\frac{10x}{x+1} = 4 \Rightarrow x = \frac{2}{3}$$

$$\frac{10x}{x+1} = 9 \Rightarrow x = 9$$

$$I = 9 \left( \int_0^{1/9} 0 dx + \int_{1/9}^{2/3} 1 dx + \int_{2/3}^9 2 dx \right)$$

$$= 155$$


---

## Question 12

Let  $y = f(x)$  be a thrice differentiable function in  $(-5, 5)$ . Let the tangents to the curve  $y = f(x)$  at  $(1, f(1))$  and  $(3, f(3))$  make angles  $\frac{\pi}{6}$  and  $\frac{\pi}{4}$ , respectively with positive  $x$ -axis. If  $27 \int_1^3 ((f'(t))^2 + 1) f''(t) dt = \alpha + \beta\sqrt{3}$  where  $\alpha, \beta$  are integers, value of  $\alpha + \beta$  equals

[30-Jan]

**Options:**

- A.
- 14
- B.
- 26
- C.
- 16
- D.
- 36

**Answer: B**

**Solution:**

$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x)$$

$$\left(\frac{dy}{dx}\right)_{(1, f(1))} = f'(1) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \Rightarrow f'(1) = \frac{1}{\sqrt{3}}$$

$$\left(\frac{dy}{dx}\right)_{(3, f(3))} = f'(3) = \tan \frac{\pi}{4} = 1 \Rightarrow f'(3) = 1$$

$$27 \int_1^3 ((f'(t))^2 + 1) f''(t) dt = \alpha + \beta\sqrt{3}$$

$$I = \int_1^3 ((f'(t))^2 + 1) f''(t) dt$$

$$f'(t) = z \Rightarrow f''(t) dt = dz$$

$$z = f'(3) = 1$$

$$z = f'(1) = \frac{1}{\sqrt{3}}$$

$$I = \int_{1/\sqrt{3}}^1 (z^2 + 1) dz = \left( \frac{z^3}{3} + z \right)_{1/\sqrt{3}}^1$$

$$= \left( \frac{1}{3} + 1 \right) - \left( \frac{1}{3} \cdot \frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \right)$$

$$= \frac{4}{3} - \frac{10}{9\sqrt{3}} = \frac{4}{3} - \frac{10}{27}\sqrt{3}$$

$$\alpha + \beta\sqrt{3} = 27 \left( \frac{4}{3} - \frac{10}{27}\sqrt{3} \right) = 36 - 10\sqrt{3}$$

$$\alpha = 36, \beta = -10$$

$$\alpha + \beta = 36 - 10 = 26$$

## Question13

Let  $f : R \rightarrow R$  be defined  $f(x) = ae^{2x} + be^x + cx$ . If  $f(0) = -1, f'(\log_e 2) = 21$  and  $\int_0^{\log_e 4} \log_e 4(f(x) - cx) dx = \frac{39}{2}$ , then the value of  $|a + b + c|$  equals :

[30-Jan

Options:

A.

16

B.

10

C.

12

D.

8

Answer: D

Solution:

$$f(x) = ae^{2x} + be^x + cx$$

$$f(0) = -1$$

$$a + b = -1$$

$$f'(x) = 2ae^{2x} + be^x + c$$

$$f'(\ln 2) = 21$$

$$8a + 2b + c = 21$$

$$\int_0^{\ln 4} (ae^{2x} + be^x) dx = \frac{39}{2}$$

$$\left[ \frac{ae^{2x}}{2} + be^x \right]_0^{\ln 4} = \frac{39}{2} \Rightarrow 8a + 4b - \frac{a}{2} - b = \frac{39}{2}$$

$$15a + 6b = 39$$

$$15a - 6a - 6 = 39$$

$$9a = 45 \Rightarrow a = 5$$

$$b = -6$$

$$c = 21 - 40 + 12 = -7$$

$$a + b + c = -8$$

$$|a + b + c| = 8$$

## Question14

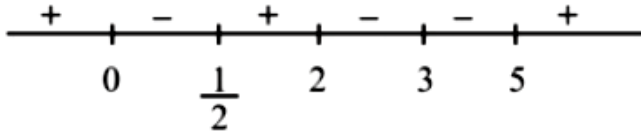
Let  $S = (-1, \infty)$  and  $f : S \rightarrow \mathbb{R}$  be defined as  $f(x) = \int_{-1}^x (e^t - 1)^{11}(2t - 1)^5(t - 2)^7(t - 3)^{12}(2t - 10)^{61} dt$  Let  $p =$  Sum of square values of  $x$ , where  $f(x)$  attains local maxima on  $S$ . and  $q =$  Sum of the values of  $x$ , where  $f(x)$  attains local minima. Then, the value of  $p^2 + 2q$  is \_\_\_\_\_

[31-Jan]

**Answer: 27**

**Solution:**

$$f'(x) = (e^x - 1)^{11}(2x - 1)^5(x - 2)^7(x - 3)^{12}(2x - 10)^{61}$$



Local minima at  $x = \frac{1}{2}, x = 5$

Local maxima at  $x = 0, x = 2$

$$\therefore p = 0 + 4 = 4, q = \frac{1}{2} + 5 = \frac{11}{2}$$

$$\text{Then } p^2 + 2q = 16 + 11 = 27$$

## Question 15

If the integral  $525 \int_0^{\frac{\pi}{2}} \sin 2x \cos \frac{11}{2}x \left(1 + \cos \frac{5}{2}x\right)^{\frac{1}{2}} dx$  is equal to  $(n\sqrt{2} - 64)$ , then  $n$  is equal to \_\_\_\_\_

[31-Jan-2024 Shift 1]

**Answer: 176**

**Solution:**

$$I = \int_0^{\frac{\pi}{2}} \sin 2x \cdot (\cos x)^{\frac{11}{2}} \left(1 + (\cos x)^{\frac{5}{2}}\right)^{\frac{1}{2}} dx$$

Put  $\cos x = t^2 \Rightarrow \sin x dx = -2t dt$

$$\therefore I = 4 \int_0^1 t^2 \cdot t^{11} \sqrt{1+t^5} (t) dt$$

$$I = 4 \int_0^1 t^{14} \sqrt{1+t^5} dt$$

Put  $1+t^5 = k^2$

$$\Rightarrow 5t^4 dt = 2k dk$$

$$\therefore I = 4 \cdot \int_1^{\sqrt{2}} (k^2-1)^2 \cdot k \cdot \frac{2k}{5} dk$$

$$I = \frac{8}{5} \int_1^{\sqrt{2}} k^6 - 2k^4 + k^2 dk$$

$$I = \frac{8}{5} \left[ \frac{k^7}{7} - \frac{2k^5}{5} + \frac{k^3}{3} \right]_1^{\sqrt{2}}$$

$$I = \frac{8}{5} \left[ \frac{8\sqrt{2}}{7} - \frac{8\sqrt{2}}{5} + \frac{2\sqrt{2}}{3} - \frac{1}{7} + \frac{2}{5} - \frac{1}{3} \right]$$

$$I = \frac{8}{5} \left[ \frac{22\sqrt{2}}{105} - \frac{8}{105} \right]$$

$$\therefore 525 \cdot I = 176\sqrt{2} - 64$$

## Question 16

Let  $f, g : (0, \infty) \rightarrow \mathbb{R}$  be two functions defined by  $f(x) = \int_{-x}^x (|t| - t^2)e^{-t^2} dt$  and  $g(x) = \int_0^x x^2 t^{1/2} e^{-t} dt$ . Then the value of  $(f(\sqrt{\log_e 9}) + g(\sqrt{\log_e 9}))$  is equal to

[31-Jan

Options:

A.

6

B.

9

C.

8

D.

10

Answer: C

Solution:

$$f(x) = \int_{-x}^x (|t| - t^2)e^{-t^2} dt$$

$$\Rightarrow f'(x) = 2 \cdot (|x| - x^2)e^{-x^2} \dots \dots \dots (1)$$

$$g(x) = \int_0^{x^2} t^{\frac{1}{2}} e^{-t} dt$$

$$g'(x) = xe^{-x^2}(2x) - 0$$

$$f'(x) + g'(x) = 2xe^{-x^2} - 2x^2e^{-x^2} + 2x^2e^{-x^2}$$

Integrating both sides w.r.t. x

$$f(x) + g(x) = \int_0^a 2xe^{-x^2} dx$$

$$x^2 = t$$

$$\Rightarrow \int_0^{\sqrt{a}} e^{-t} dt = [-e^{-t}]_0^{\sqrt{a}}$$

$$= -e^{(\log_e(9)^{-1})} + 1$$

$$\Rightarrow 9(f(x) + g(x)) = \left(1 - \frac{1}{9}\right)9 = 8$$

## Question 17

$$\left| \frac{120}{\pi^3} \int_0^{\frac{\pi}{2}} \frac{x^2 \sin x \cos x}{\sin^4 x + \cos^4 x} dx \right| \text{ is equal to } \underline{\hspace{2cm}}$$

[31-Jan-2024 Shift 2]

Answer: 15

Solution:

$$\begin{aligned}
& \int_0^{\pi} \frac{x^2 \sin x \cdot \cos x}{\sin^4 x + \cos^4 x} dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x} (x^2 - (\pi - x)^2) dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin x \cdot \cos x (2\pi x - \pi^2)}{\sin^4 x + \cos^4 x} dx \\
&= 2\pi \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx - \pi^2 \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx \\
&= 2\pi \cdot \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos^4 x}{\sin^4 x + \cos^4 x} dx - \pi^2 \int_0^{\frac{\pi}{2}} \frac{\sin x \cos^4 x}{\sin^4 x + \cos^4 x} dx \\
&= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx \\
&= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x dx}{1 - 2\sin^2 x \times \cos^2 x} \\
&= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2 - \sin^2 2x} dx \\
&= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1 + \cos^2 2x} dx
\end{aligned}$$

Let  $\cos 2x = t$

## Question 18

The value of the integral  $\int_0^{\frac{\pi}{4}} \frac{x dx}{\sin^4(2x) + \cos^4(2x)}$  equals :

[1-Feb-2024 Shift 1]

Options:

A.

$$\frac{\sqrt{2}\pi^2}{8}$$

B.

$$\frac{\sqrt{2}\pi^2}{16}$$

C.

$$\frac{\sqrt{2}\pi^2}{32}$$

D.



$$\frac{\sqrt{2}\pi^2}{64}$$

**Answer: C**

**Solution:**

$$\int_0^{\frac{\pi}{4}} \frac{x \, dx}{\sin^4(2x) + \cos^4(2x)}$$

Let  $2x = t$  then  $dx = \frac{1}{2} dt$

$$I = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{t \, dt}{\sin^4 t + \cos^4 t}$$

$$I = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - t\right) dt}{\sin^4\left(\frac{\pi}{2} - t\right) + \cos^4\left(\frac{\pi}{2} - t\right)}$$

$$I = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} dt}{\sin^4 t + \cos^4 t} - I$$

$$2I = \frac{\pi}{8} \int_0^{\frac{\pi}{2}} \frac{dt}{\sin^4 t + \cos^4 t}$$

$$2I = \frac{\pi}{8} \int_0^{\frac{\pi}{2}} \frac{\sec^4 t \, dt}{\tan^4 t + 1}$$

Let  $\tan = y$  then  $\sec^2 t \, dt = dy$

$$2I = \frac{\pi}{8} \int_0^{\infty} \frac{(1+y^2)dy}{1+y^4}$$

$$= \frac{\pi}{16} \int_0^{\infty} \frac{1 + \frac{1}{y^2}}{y^2 + \frac{1}{y^2}} dy$$

Put  $y - \frac{1}{y} = p$

$$I = \frac{\pi}{16} \int_{-\infty}^{\infty} \frac{dp}{p^2 + (\sqrt{2})^2}$$

$$= \frac{\pi}{16\sqrt{2}} \left[ \tan^{-1}\left(\frac{p}{\sqrt{2}}\right) \right]_{-\infty}^{\infty}$$

$$I = \frac{\pi^2}{16\sqrt{2}}$$

## Question 19

If  $\int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2} \cos x \, dx}{(1 + e^{\sin x})(1 + \sin^4 x)} = \alpha\pi + \beta \log_e(3 + 2\sqrt{2})$ , where  $\alpha, \beta$  are integers, then  $\alpha^2 + \beta^2$  equals

[1-Feb-2024 Shift 1]

**Answer: None**

### Solution:

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2} \cos x}{(1 + e^{\sin x})(1 + \sin^4 x)} dx \dots\dots\dots(1)$$

Apply king

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2} \cos x (e^{\sin x})}{(1 + e^{\sin x})(1 + \sin^4 x)} dx \dots\dots\dots(2)$$

adding (1) & (2)

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2} \cos x}{1 + \sin^4 x} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{8\sqrt{2} \cos x}{1 + \sin^4 x} dx,$$

$$\sin x = t$$

$$I = \int_0^1 \frac{8\sqrt{2}}{1 + t^4} dx$$

$$I = 4\sqrt{2} \int_0^1 \left( \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} - \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} \right) dt$$

$$I = 4\sqrt{2} \int_0^1 \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} - \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 - 2} dt$$

$$\text{Let } t - \frac{1}{t} = z \text{ \& } t + \frac{1}{t} = k$$

$$= 4\sqrt{2} \left[ \int_{-\infty}^0 \frac{dz}{z^2 + 2} - \int_{\infty}^2 \frac{dk}{k^2 - 2} \right]$$

$$= 4\sqrt{2} \left[ \frac{1}{\sqrt{2}} \tan^{-1} \frac{z}{\sqrt{2}} \right]_{-\infty}^0 - \left[ \frac{1}{2\sqrt{2}} \ln \left( \frac{k - \sqrt{2}}{k + \sqrt{2}} \right) \right]_{\infty}^2$$

$$= 4\sqrt{2} \left[ \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \left[ \ln \frac{2 - \sqrt{2}}{2 + \sqrt{2}} \right] \right]$$

$$= 2\pi + 2 \ln(3 + 2\sqrt{2})$$

$$\alpha = 2$$

$$\beta = 2$$

---

### Question20

The value of  $\int_0^1 (2x^3 - 3x^2 - x + 1)^{\frac{1}{3}} dx$  is equal to:

[1-Feb-2024 Shift 2]

Options:

- A.
- 0
- B.
- 1
- C.
- 2
- D.
- 1

**Answer: A**

**Solution:**

$$I = \int_0^1 (2x^3 - 3x^2 - x + 1)^{\frac{1}{3}} dx$$

Using  $\int_0^{2a} f(x) dx$  where  $f(2a-x) = -f(x)$

Here  $f(1-x) = f(x)$

$\therefore I = 0$

## Question21

If  $\int_0^{\frac{\pi}{3}} \cos^4 x dx = a\pi + b\sqrt{3}$ , where  $a$  and  $b$  are rational numbers, then  $9a + 8b$  is equal to :

[1-Feb-2024 Shift 2]

**Options:**

- A.
- 2
- B.
- 1
- C.
- 3
- D.
- 3/2

**Answer: A**

**Solution:**



$$\begin{aligned}
& \int_0^{\pi/3} \cos^4 x \, dx \\
&= \int_0^{\pi/3} \left( \frac{1 + \cos 2x}{2} \right)^2 dx \\
&= \frac{1}{4} \int_0^{\pi/3} (1 + 2 \cos 2x + \cos^2 2x) dx \\
&= \frac{1}{4} \left[ \int_0^{\pi/3} dx + 2 \int_0^{\pi/3} \cos 2x \, dx + \int_0^{\pi/3} \frac{1 + \cos 4x}{2} dx \right] \\
&= \frac{1}{4} \left[ \frac{\pi}{3} + (\sin 2x)_0^{\pi/3} + \frac{1}{2} \left( \frac{\pi}{3} \right) + \frac{1}{8} (\sin 4x)_0^{\pi/3} \right] \\
&= \frac{1}{4} \left[ \frac{\pi}{3} + (\sin 2x)_0^{\pi/3} + \frac{1}{2} \left( \frac{\pi}{3} \right) + \frac{1}{8} (\sin 4x)_0^{\pi/3} \right] \\
&= \frac{1}{4} \left[ \frac{\pi}{2} + \frac{\sqrt{3}}{2} + \frac{1}{8} \times \left( -\frac{\sqrt{3}}{2} \right) \right] \\
&= \frac{\pi}{2} + \frac{7\sqrt{3}}{64} \\
\therefore a &= \frac{1}{8}; b = \frac{7}{64} \\
\therefore 9a + 8b &= \frac{9}{8} + \frac{7}{8} = 2
\end{aligned}$$

## Question22

Let  $f : (0, \infty) \rightarrow \mathbb{R}$  and  $F(x) = \int_0^x f(t) \, dt$ . If  $F(x^2) = x^4 + x^5$ , then  $\sum_{r=1}^{12} f(r^2)$  is equal to :

[1-Feb-2024 Shift 2]

Answer: 219

Solution:

$$F(x) = \int_0^x t \cdot f(t) dt$$

$$F^1(x) = xf(x)$$

Given  $F(x^2) = x^4 + x^5$ , let  $x^2 = t$

$$F(t) = t^2 + t^{5/2}$$

$$F'(t) = 2t + 5/2t^{3/2}$$

$$t \cdot f(t) = 2t + 5/2t^{3/2}$$

$$f(t) = 2 + 5/2t^{1/2}$$

$$\sum_{r=1}^{12} f(r^2) = \sum_{r=1}^{12} 2 + \frac{5}{2}r$$

$$= 24 + 5/2 \left[ \frac{12(13)}{2} \right]$$

$$= 219$$

## Question23

The value of  $\frac{8}{\pi} \int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}} dx$  is \_\_\_\_\_

[24-Jan-2023 Shift 1]

Answer: 2

Solution:

$$I = \frac{8}{\pi} \int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}} dx \dots \dots (1)$$

$$\text{Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \frac{8}{\pi} \int_0^{\frac{\pi}{2}} \frac{(\sin x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}} dx \dots \dots (2)$$

Adding (1)&(2)

$$II = \frac{8}{\pi} \int_0^{\frac{\pi}{2}} 1 dx$$

## Question24

$\frac{3\sqrt{3}}{4} \int_{\frac{3\sqrt{2}}{4}}^{\frac{48}{\sqrt{9-4x^2}}} dx$  is equal to

[24-Jan-2023 Shift 2]

Options:

- A.  $\frac{\pi}{3}$   
 B.  $\frac{\pi}{2}$   
 C.  $\frac{\pi}{6}$   
 D.  $2\pi$

**Answer: D**

**Solution:**

**Solution:**

$$\frac{3\sqrt{3}}{4} \int \frac{48}{3\sqrt{2} \sqrt{9-4x^2}} dx$$

We have  $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C$

Hence  $\frac{3\sqrt{3}}{4} \int \frac{48}{3\sqrt{2} \sqrt{9-4x^2}} dx = \frac{48}{2} \times \left[ \sin^{-1} \frac{2x}{3} \right] \frac{3\sqrt{3}}{4}$

$$= 24 \times \left[ \sin^{-1} \left( \frac{2}{3} \times \frac{3\sqrt{3}}{4} \right) - \sin^{-1} \left( \frac{2}{3} \times \frac{3\sqrt{2}}{4} \right) \right]$$

$$= 24 \times \left[ \sin^{-1} \frac{\sqrt{3}}{2} - \sin^{-1} \frac{1}{\sqrt{2}} \right]$$

$$= 24 \times \left( \frac{\pi}{3} - \frac{\pi}{4} \right)$$

$$= 24 \times \frac{\pi}{12} = 2\pi$$

## Question 25

Let  $f$  be a differentiable function defined on  $\left[0, \frac{\pi}{2}\right]$  such that  $f(x) > 0$  and

$$f(x) + \int_0^x f(t) \sqrt{1 - (\log_e f(t))^2} dt = e, \forall x \in \left[0, \frac{\pi}{2}\right].$$

Then  $\left(6 \log_e f\left(\frac{\pi}{6}\right)\right)^2$  is equal to \_\_\_\_

[24-Jan-2023 Shift 2]

**Answer: 27**

**Solution:**

$$f(x) + \int_0^x f(t) \sqrt{1 - (\log_e f(t))^2} dt = e$$

$$\Rightarrow f(0) = e$$

$$f'(x) + f(x) \sqrt{1 - (\ln f(x))^2} = 0$$

$$f(x) = y$$

$$\frac{dy}{dx} = -y \sqrt{1 - (\ln y)^2}$$

$$\int \frac{dy}{y \sqrt{1 - (\ln y)^2}} = -\int dx$$

Put  $\ln y = t$

$$\int \frac{dt}{\sqrt{1-t^2}} = -x + C$$

$$\sin^{-1} t = -x + C \Rightarrow \sin^{-1}(\ln y) = -x + C$$

$$\sin^{-1}(\ln f(x)) = -x + C$$

$$f(0) = e$$

$$\Rightarrow \frac{\pi}{2} = C$$

$$\Rightarrow \sin^{-1}(\ln f(x)) = -x + \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\ln f\left(\frac{\pi}{6}\right)\right) = \frac{-\pi}{6} + \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\ln f\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{3}$$

$$\Rightarrow \ln f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \text{ we need } \left(6 \times \frac{\sqrt{3}}{2}\right)^2 = 27.$$

---

## Question26

The minimum value of the function  $f(x) = \int_0^x e^{-t} dt$  is is  
[25-Jan-2023 Shift 1]

Options:

A.  $2(e-1)$

B.  $2e-1$

C. 2

D.  $e(e-1)$

Answer: A

Solution:

Solution:

For  $x \leq 0$

$$f(x) = \int_0^x e^{-t} dt = e^{-x}(e^2 - 1)$$

For  $0 < x < 2$

$$f(x) = \int_0^x e^{-t} dt + \int_x^2 e^{-t} dt = e^{-x} + e^{-2} - 2$$

For  $x \geq 2$

$$f(x) = \int_0^2 e^{-t} dt = e^{-2}(e^2 - 1)$$

For  $x \leq 0$ ,  $f(x)$  is  $\downarrow$  and  $x \geq 2$ ,  $f(x)$  is  $\uparrow$

$\therefore$  Minimum value of  $f(x)$  lies in  $x \in (0, 2)$

Applying A.M  $\geq$  G.M

minimum value of  $f(x)$  is  $2(e-1)$

---

## Question27

The value of

$$\lim_{n \rightarrow \infty} \frac{1+2-3+4+5-6+\dots+(3n-2)+(3n-1)-3n}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$$

is :

[25-Jan-2023 Shift 1]

Options:

A.  $\frac{\sqrt{2}+1}{2}$

B.  $3(\sqrt{2} + 1)$

C.  $\frac{3}{2}(\sqrt{2} + 1)$

D.  $\frac{3}{2\sqrt{2}}$

**Answer: C**

**Solution:**

**Solution:**

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{0+3+6+9+\dots+n \text{ terms}}{\sqrt{2n^4+4n+3} - \sqrt{n^4+5n+4}} \\ \lim_{n \rightarrow \infty} \frac{3n(n-1)}{2(\sqrt{2n^4+4n+3} - \sqrt{n^4+5n+4})} \\ = \frac{3}{2(\sqrt{2}-1)} = \frac{3}{2}(\sqrt{2}+1) \end{aligned}$$


---

## Question28

The integral  $16 \int_1^2 \frac{dx}{x^3(x^2+2)^2}$  is equal to  
[25-Jan-2023 Shift 2]

**Options:**

A.  $\frac{11}{6} + \log_e 4$

B.  $\frac{11}{12} + \log_e 4$

C.  $\frac{11}{12} - \log_e 4$

D.  $\frac{11}{6} - \log_e 4$

**Answer: D**

**Solution:**

**Solution:**

$$\begin{aligned} I &= 16 \int_1^2 \frac{dx}{x^3(x^2+2)^2} \\ &= 16 \int_1^2 \frac{dx}{x^3 x^4 \left(1 + \frac{2}{x^2}\right)^2} \end{aligned}$$

$$\text{Let, } 1 + \frac{2}{x^2} = t \Rightarrow \frac{-4}{x^3} dx = dt$$

$$I = -4 \int_3^{\frac{3}{2}} \frac{dt}{\left(\frac{t}{2}\right)^2 t^2}$$

$$I = -4 \int_3^{\frac{3}{2}} \left(\frac{t-1}{2}\right)^2 \frac{dt}{t^2}$$

$$I = -\frac{4}{4} \int_3^{\frac{3}{2}} \left(1 - \frac{2}{t} + \frac{1}{t^2}\right) dt$$

$$I = -1 \left[ t - 2 \ln |t| - \frac{1}{t} \right]_3^{\frac{3}{2}}$$

$$I = -1 \left[ \left( \frac{3}{2} - 2 \ln \frac{3}{2} - \frac{2}{3} \right) - \left( 3 - 2 \ln 3 - \frac{1}{3} \right) \right]$$



$$I = -1 \left[ 2 \ln 2 - \frac{11}{6} \right]$$

$$I = \frac{11}{6} - \ln 4$$

## Question29

If  $\int_{\frac{1}{3}}^3 \left| \log_e x \right| dx = \frac{m}{n} \log_e \left( \frac{n^2}{e} \right)$ , where  $m$  and  $n$  are coprime natural numbers, then  $m^2 + n^2 - 5$  is equal to

[25-Jan-2023 Shift 2]

Answer: 20

Solution:

$$\begin{aligned} \int_{\frac{1}{3}}^3 \left| \ln x \right| dx &= \int_{\frac{1}{3}}^1 (-\ln x) dx + \int_1^3 (\ln x) dx \\ &= -[x \ln x - x]_{1/3}^1 + [x \ln x - x]_1^3 \\ &= -\left[ -1 - \left( \frac{1}{3} \ln \frac{1}{3} - \frac{1}{3} \right) \right] + [3 \ln 3 - 3 - (-1)] \\ &= \left[ -\frac{2}{3} - \frac{1}{3} \ln \frac{1}{3} \right] + [3 \ln 3 - 2] \\ &= -\frac{4}{3} + \frac{8}{3} \ln 3 \\ &= \frac{4}{3} (2 \ln 3 - 1) \\ &= \frac{4}{3} \left( \ln \frac{9}{e} \right) \end{aligned}$$

$$\therefore m = 4, n = 3$$

$$\text{Now, } m^2 + n^2 - 5 = 16 + 9 - 5 = 20$$

## Question30

Let  $[x]$  denote the greatest integer  $\leq x$ . Consider the function  $f(x) = \max\{x^2, 1 + [x]\}$ . Then the value of the integral  $\int_0^2 f(x) dx$  is:

[29-Jan-2023 Shift 1]

Options:

A.  $\frac{5+4\sqrt{2}}{3}$

B.  $\frac{8+4\sqrt{2}}{3}$

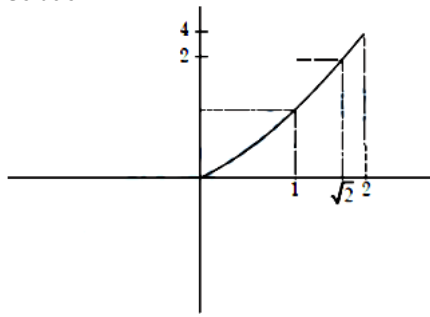
C.  $\frac{1+5\sqrt{2}}{3}$

D.  $\frac{4+5\sqrt{2}}{3}$

Answer: A

## Solution:

Solution:



$$\begin{aligned}
 A &= \int_0^1 1 \cdot dx + \int_1^{\sqrt{2}} 2 dx + \int_{\sqrt{2}}^2 x^2 dx \\
 &= 1 + 2\sqrt{2} - 2 + \frac{8}{3} - \frac{2\sqrt{2}}{3} \\
 &= \frac{5}{3} + \frac{4\sqrt{2}}{3}
 \end{aligned}$$

## Question31

Let  $f(x) = x + \frac{a}{\pi^2 - 4} \sin x + \frac{b}{\pi^2 - 4} \cos x$ ,  $x \in \mathbb{R}$  be a function which satisfies  $f(x) = x + \int_0^{\pi/2} \sin(x+y)f(y)dy$ .

Then  $(a + b)$  is equal to

[29-Jan-2023 Shift 1]

Options:

A.  $-\pi(\pi + 2)$

B.  $-2\pi(\pi + 2)$

C.  $-2\pi(\pi - 2)$

D.  $-\pi(\pi - 2)$

Answer: B

Solution:

Solution:

$$f(x) = x + \int_0^{\pi/2} (\sin x \cos y + \cos x \sin y) f(y) dy$$

$$f(x) = x + \int_0^{\pi/2} ((\cos y f(y) dy) \sin x + (\sin y f(y) dy) \cos x)$$

On comparing with

$$f(x) = x + \frac{a}{\pi^2 - 4} \sin x + \frac{b}{\pi^2 - 4} \cos x, x \in \mathbb{R} \text{ then}$$

$$\Rightarrow \frac{a}{\pi^2 - 4} = \int_0^{\pi/2} \cos y f(y) dy \dots (2)$$

$$\Rightarrow \frac{b}{\pi^2 - 4} = \int_0^{\pi/2} \sin y f(y) dy \dots (3)$$

Add (2) and (3)

$$\frac{a+b}{\pi^2 - 4} = \int_0^{\pi/2} (\sin y + \cos y) f(y) dy \dots (4)$$

$$\frac{a+b}{\pi^2 - 4} = \int_0^{\pi/2} (\sin y + \cos y) f\left(\frac{\pi}{2} - y\right) dy \dots (5)$$

Add (4) and (5)

$$\frac{2(a+b)}{\pi^2 - 4} = \int_0^{\pi/2} (\sin y + \cos y) \left( \frac{\pi}{2} + \frac{(a+b)}{\pi^2 - 4} (\sin y + \cos y) \right) dy$$

$$= \pi + \frac{a+b}{\pi^2 - 4} \left( \frac{\pi}{2} + 1 \right)$$

$$(a + b) = -2\pi(\pi + 2)$$

---

## Question32

The value of the integral  $\int_{1/2}^2 \frac{\tan^{-1}x}{x} dx$  is equal to  
[29-Jan-2023 Shift 2]

Options:

- A.  $\pi \log_e 2$
- B.  $\frac{1}{2} \log_e 2$
- C.  $\frac{\pi}{4} \log_e 2$
- D.  $\frac{\pi}{2} \log_e 2$

Answer: D

Solution:

Solution:

$$I = \int_{1/2}^2 \frac{\tan^{-1}x}{x} dx \dots (i)$$

$$\text{Put } x = \frac{1}{t} \quad dx = -\frac{1}{t^2} dt$$

$$I = -\int_2^{1/2} \frac{\tan^{-1}\frac{1}{t}}{\frac{1}{t}} \cdot \frac{1}{t^2} dt = -\int_2^{1/2} \frac{\tan^{-1}\frac{1}{t}}{t} dt$$

$$I = \int_{1/2}^2 \frac{\cot^{-1}t}{t} dt = \int_{1/2}^2 \frac{\cot^{-1}x}{x} dx \dots (ii)$$

Add both equation

$$2I = \int_{1/2}^2 \frac{\tan^{-1}x + \cot^{-1}x}{x} dx = \frac{\pi}{2} \int_{1/2}^2 \frac{dx}{x} = \frac{\pi}{2} (\ln 2)_{1/2}^2$$
$$= \frac{\pi}{2} \left( \ln 2 - \ell \ln \frac{1}{2} \right) = \pi \ell \ln 2$$

$$I = \frac{\pi}{2} \ell \ln 2$$

---

## Question33

The value of the integral  $\int_1^2 \left( \frac{t^4+1}{t^6+1} \right) dt$  is :

[29-Jan-2023 Shift 2]

Options:

- A.  $\tan^{-1} \frac{1}{2} + \frac{1}{3} \tan^{-1} 8 - \frac{\pi}{3}$
- B.  $\tan^{-1} 2 - \frac{1}{3} \tan^{-1} 8 + \frac{\pi}{3}$
- C.  $\tan^{-1} 2 + \frac{1}{3} \tan^{-1} 8 - \frac{\pi}{3}$
- D.  $\tan^{-1} \frac{1}{2} - \frac{1}{3} \tan^{-1} 8 + \frac{\pi}{3}$

Answer: C

Solution:

$$\begin{aligned} I &= \int_1^2 \left( \frac{t^4+1}{t^6+1} \right) dt \\ &= \int_1^2 \frac{(t^4+1-t^2)+t^2}{(t^2+1)(t^4-t^2+1)} dt \\ &= \int_1^2 \left( \frac{1}{t^2+1} + \frac{t^2}{t^6+1} \right) dt \\ &= \int_1^2 \left( \frac{1}{t^2+1} + \frac{1}{3} \frac{3t^2}{(t^3)^2+1} \right) dt \\ &= \tan^{-1}(t) + \frac{1}{3} \tan^{-1}(t^3) \Big|_1^2 \\ &= (\tan^{-1}(2) - \tan^{-1}(1)) + \frac{1}{3} (\tan^{-1}(2^3) - \tan^{-1}(1^3)) \\ &= \tan^{-1}(2) + \frac{1}{3} \tan^{-1}(8) - \frac{\pi}{3} \end{aligned}$$

---

### Question34

If  $\lfloor t \rfloor$  denotes the greatest integer  $\leq t$ , then the value of  $\int_1^2 \frac{3(e-1)^2}{e} x^2 e^{\lfloor x \rfloor + \lfloor x^3 \rfloor} dx$  is :  
[30-Jan-2023 Shift 1]

Options:

- A.  $e^9 - e$
- B.  $e^8 - e$
- C.  $e^7 - 1$
- D.  $e^8 - 1$

Answer: B

Solution:

$$\begin{aligned} &\int_1^2 x^2 e^{\lfloor x^3 \rfloor + 1} dx \\ &x^3 = t \\ &3x^2 dx = dt \\ &= \frac{e}{3} \int_1^8 e^{\lfloor t \rfloor} dt \\ &= \frac{e}{3} \left\{ \int_1^2 e dt + \int_2^3 e^2 dt + \dots + \int_7^8 e^7 dt \right\} \\ &= \frac{e}{3} (e + e^2 + \dots + e^7) \\ &= \frac{e^2}{3} (1 + e + \dots + e^6) = \frac{e^2}{3} \frac{(e^7 - 1)}{(e - 1)} \\ &\frac{3(e-1)^2}{e} \int_1^2 x^2 \times e^{\lfloor x \rfloor + \lfloor x^3 \rfloor} dx = \frac{3}{e} (e-1) \times \frac{e^2}{3} \frac{(e^7 - 1)}{(e - 1)} \\ &= e(e^7 - 1) \\ &= e^8 - e \end{aligned}$$

---

### Question35

$\lim_{x \rightarrow 0} \frac{48}{x^4} \int_0^x \frac{t^3}{t^6+1} dt$  is equal to \_\_\_\_\_.

[30-Jan-2023 Shift 1]

Answer: 12

Solution:

$$48 \lim_{x \rightarrow 0} \frac{\int_0^x \frac{t^3}{t^6+1} dt}{x^4} \left( \frac{0}{0} \right)$$

Applying L'Hospitals Rule

$$48 \lim_{x \rightarrow 0} \frac{x^3}{x^6+1} \times \frac{1}{4x^3} \\ = 12$$

### Question36

$$\lim_{n \rightarrow \infty} \frac{3}{n} \left\{ 4 + \left( 2 + \frac{1}{n} \right)^2 + \left( 2 + \frac{2}{n} \right)^2 + \dots + \left( 3 - \frac{1}{n} \right)^2 \right\}$$

is equal to

[30-Jan-2023 Shift 2]

Options:

A. 12

B.  $\frac{19}{3}$

C. 0

D. 19

Answer: D

Solution:

$$\lim_{n \rightarrow \infty} \frac{3}{n} \sum_{r=0}^{n-1} \left( 2 + \frac{r}{n} \right)^2 \\ = 3 \int_0^1 (2+x)^2 dx = 27 - 8 = 19$$

### Question37

The value of  $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(2+3\sin x)}{\sin x(1+\cos x)} dx$  is equal to

[31-Jan-2023 Shift 1]

Options:

A.  $\frac{7}{2} - \sqrt{3} - \log_e \sqrt{3}$

B.  $-2 + 3\sqrt{3} + \log_e \sqrt{3}$

C.  $\frac{10}{3} - \sqrt{3} + \log_e \sqrt{3}$

D.  $\frac{10}{3} - \sqrt{3} - \log_e \sqrt{3}$

**Answer: C**

**Solution:**

**Solution:**

$$\int_{\pi/3}^{\pi/2} \left( \frac{2 + 3\sin x}{\sin x(1 + \cos x)} \right) dx = 2 \int_{\pi/3}^{\pi/2} \frac{dx}{\sin x + \sin x \cos x} + 3$$

$$3 \int_{\pi/3}^{\pi/2} \frac{dx}{1 + \cos x}$$

$$\int_{\pi/3}^{\pi/2} \frac{dx}{1 + \cos x} = \int_{\pi/3}^{\pi/2} \frac{1 - \cos x}{\sin^2 x} dx$$

$$= \int_{\pi/3}^{\pi/2} (\operatorname{cosec}^2 x - \cot x \operatorname{cosec} x) dx$$

$$= (\operatorname{cosec} x - \cot x) \Big|_{\pi/3}^{\pi/2} = (1) - \left( \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right) = 1 - \frac{1}{\sqrt{3}}$$

$$\int_{\pi/3}^{\pi/2} \frac{dx}{\sin x(1 + \cos x)} =$$

$$\int \frac{dx}{(2 \tan x/2)(1 + 1 - \tan^2 x/2)}$$

$$= \int \frac{(1 + \tan^2 x/2) \sec^2 x/2 dx}{2 \tan x/2}$$

$$\tan x/2 = t$$

$$\frac{1}{2} \int \left( \frac{1+t^2}{t} \right) dt = \frac{1}{2} \left[ \ln t + \frac{t^2}{2} \right] \Big|_{\frac{1}{\sqrt{3}}}^1$$

$$= \frac{1}{2} \left[ \left( 0 + \frac{1}{2} \right) - \left( \ln \frac{1}{\sqrt{3}} + \frac{1}{6} \right) \right] = \left( \frac{1}{3} + \ln \sqrt{3} \right) \frac{1}{2}$$

$$= \left( \frac{1}{6} + \frac{1}{2} \ln \sqrt{3} \right) dt$$

$$2 \left( \frac{1}{6} + \frac{1}{2} \ln \sqrt{3} \right) + 3 \left( 1 - \frac{1}{\sqrt{3}} \right)$$

$$= \frac{1}{3} + \ln \sqrt{3} + 3 - \sqrt{3} = \frac{10}{3} + \ln \sqrt{3} - \sqrt{3}$$

## Question 38

Let a differentiable function  $f$  satisfy  $f(x) + \int_3^x \frac{f(t)}{t} dt = \sqrt{x+1}$ ,  $x \geq 3$ . Then  $12f(8)$  is equal to:  
[31-Jan-2023 Shift 1]

**Options:**

A. 34

B. 19

C. 17

D. 1

**Answer: C**

**Solution:**

Differentiate w.r.t.  $x$

$$f'(x) + \frac{f(x)}{x} = \frac{1}{2\sqrt{x+1}}$$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$xf(x) = \int \frac{x}{2\sqrt{x+1}} dx$$

$$x+1 = t^2 \\ = \int \frac{t^2-1}{2t} 2t dt$$

$$xf(x) = \frac{t^3}{3} - t + c$$

$$xf(x) = \frac{(x+1)^{3/2}}{3} - \sqrt{x+1} + c$$

Also putting  $x = 3$  in given equation  $f(3) + 0 = \sqrt{4} f(3) = 2$

$$\Rightarrow C = 8 - \frac{8}{3} = \frac{16}{3}$$

$$f(x) = \frac{\frac{(x+1)^{3/2}}{3} - \sqrt{x+1} + \frac{16}{3}}{x}$$

$$f(8) = \frac{9 - 3 + \frac{16}{3}}{8} = \frac{34}{24}$$

$$\Rightarrow 12f(8) = 17$$

## Question39

Let  $\alpha \in (0, 1)$  and  $\beta = \log_e(1 - \alpha)$ . Let  $P_n(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n}$ ,  $x \in (0, 1)$ .

Then the integral  $\int_0^\alpha \frac{t^{50}}{1-t} dt$  is equal to

[31-Jan-2023 Shift 1]

Options:

A.  $\beta - P_{50}(\alpha)$

B.  $-(\beta + P_{50}(\alpha))$

C.  $P_{50}(\alpha) - \beta$

D.  $\beta + P_{50}(\alpha)$

Answer: B

Solution:

$$\begin{aligned} \int_0^\alpha \frac{t^{50} - 1 + 1}{1-t} dt &= -\int_0^\alpha (1+t+\dots+t^{49}) dt + \int_0^\alpha \frac{1}{1-t} dt \\ &= -\left(\frac{\alpha^{50}}{50} + \frac{\alpha^{49}}{49} + \dots + \frac{\alpha^1}{1}\right) + \left(\frac{\ln(1-f)}{-1}\right)_0^\alpha \\ &= -P_{50}(\alpha) - \ln(1-\alpha) \\ &= -P_{50}(\alpha) - \beta \end{aligned}$$

## Question40

Let  $\alpha > 0$ . If  $\int_0^\alpha \frac{x}{\sqrt{x+\alpha}-\sqrt{x}} dx = \frac{16+20\sqrt{2}}{15}$ , then  $\alpha$  is equal to :

[31-Jan-2023 Shift 2]

Options:

A. 2

B. 4

C.  $\sqrt{2}$

D.  $2\sqrt{2}$

**Answer: A**

**Solution:**

**Solution:**

After rationalising

$$\int_0^{\alpha} \frac{x}{\alpha} (\sqrt{x+\alpha} + \sqrt{x})$$

$$\frac{1}{\alpha} \left[ \frac{2}{5}(x+\alpha)^{5/2} - \alpha \frac{2}{3}(x+\alpha)^{3/2} + \frac{2}{5}x^{5/2} \right] \Big|_0^{\alpha}$$

$$= \frac{1}{\alpha} \left( \frac{5}{2}(2\alpha)^{5/2} - \frac{2\alpha}{3}(2\alpha)^{3/2} + \frac{2}{5}\alpha^{5/2} - \frac{2}{5}\alpha^{5/2} + \frac{2}{3}\alpha^{5/2} \right)$$

$$= \frac{1}{\alpha} \left( \frac{2^{7/2}\alpha^{5/2}}{5} - \frac{2^{5/2}\alpha^{5/2}}{3} + \frac{2}{3}\alpha^{5/2} \right)$$

$$= \alpha^{3/2} \left( \frac{2^{7/2}}{5} - \frac{2^{5/2}}{3} + \frac{2}{3} \right)$$

$$= \frac{\alpha^{3/2}}{15} (24\sqrt{2} - 20\sqrt{2} + 10) = \frac{\alpha^{3/2}}{15} (4\sqrt{2} + 10)$$

Now,

$$\frac{\alpha^{3/2}}{15} (4\sqrt{2} + 10) = \frac{16 + 20\sqrt{2}}{15}$$

$$\Rightarrow \alpha = 2$$

## Question41

If  $\phi(x) = \frac{1}{\sqrt{x}} \int_{\frac{\pi}{4}}^x (4\sqrt{2}\sin t - 3\phi'(t)) dt$ ,  $x > 0$  then  $\phi' \left( \frac{\pi}{4} \right)$  is equal to :

[31-Jan-2023 Shift 2]

**Options:**

A.  $\frac{8}{\sqrt{\pi}}$

B.  $\frac{4}{6 + \sqrt{\pi}}$

C.  $\frac{8}{6 + \sqrt{\pi}}$

D.  $\frac{4}{6 - \sqrt{\pi}}$

**Answer: C**

**Solution:**

**Solution:**

$$\phi'(x) = \frac{1}{\sqrt{x}} [(4\sqrt{2}\sin x - 3\phi'(x)) \cdot 1 - 0] - \frac{1}{2}x^{-3/2}$$

$$\int_{\frac{\pi}{4}}^x (4\sqrt{2}\sin t - 3\phi'(t)) dt$$

$$\phi' \left( \frac{\pi}{4} \right) = \frac{2}{\sqrt{\pi}} \left[ 4 - 3\phi' \left( \frac{\pi}{4} \right) \right] + 0$$

$$\left( 1 + \frac{6}{\sqrt{\pi}} \right) \phi' \left( \frac{\pi}{4} \right) = \frac{8}{\sqrt{\pi}}$$

$$\phi' \left( \frac{\pi}{4} \right) = \frac{8}{\sqrt{\pi} + 6}$$



## Question42

Let  $f(x) = \int \frac{2x}{(x^2+1)(x^2+3)} dx$ .

If  $f(3) = \frac{1}{2}(\log_e 5 - \log_e 6)$ , then  $f(4)$  is equal to  
[25-Jan-2023 Shift 1]

Options:

A.  $\frac{1}{2}(\log_e 17 - \log_e 19)$

B.  $\log_e 17 - \log_e 18$

C.  $\frac{1}{2}(\log_e 19 - \log_e 17)$

D.  $\log_e 19 - \log_e 20$

Answer: A

Solution:

Solution:

Put  $x^2 = t$

$$\int \frac{dt}{(t+1)(t+3)} = \frac{1}{2} \int \left( \frac{1}{t+1} - \frac{1}{t+3} \right) dt$$

$$f(x) = \frac{1}{2} \ln \left( \frac{x^2+1}{x^2+3} \right) + C$$

$$f(3) = \frac{1}{2} (\ln 10 - \ln 12) + C$$

$$\Rightarrow C = 0$$

$$f(4) = \frac{1}{2} \ln \left( \frac{17}{19} \right)$$

## Question43

If  $\int \sqrt{\sec 2x - 1} dx = \alpha \log_e \left| \cos 2x + \beta + \sqrt{\cos 2x \left( 1 + \cos \frac{1}{\beta} x \right)} \right|$

+ constant, then  $\beta - \alpha$  is equal to \_\_\_\_\_.

[30-Jan-2023 Shift 2]

Answer: 1

Solution:

$$\int \sqrt{\sec 2x - 1} dx = \int \sqrt{\frac{1 - \cos 2x}{\cos 2x}} dx$$

$$= \sqrt{2} \int \frac{\sin x}{\sqrt{2\cos^2 x - 1}} dx$$

$$\text{put } \cos x = t \Rightarrow -\sin x dx = dt$$

$$= -\sqrt{2} \int \frac{dt}{\sqrt{2t^2 - 1}}$$

$$= -\ln \left| \sqrt{2} \cos x + \sqrt{\cos 2x} \right| + c$$

$$= -\frac{1}{2} \ln \left| 2\cos^2 x + \cos 2x + 2\sqrt{\cos 2x} \cdot \sqrt{2} \cos x \right| + c$$



$$= -\frac{1}{2} \ln \left| \cos^{2x} + \frac{1}{2} + \sqrt{\cos 2x} \cdot \sqrt{1 + \cos 2x} \right| + c$$

$$\because \beta = \frac{1}{2}, \alpha = -\frac{1}{2} \Rightarrow \beta - \alpha = 1$$

## Question44

$\lim_{n \rightarrow \infty} \left( \frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \dots + \frac{1}{2n} \right)$  is equal to :-

[1-Feb-2023 Shift 1]

Options:

A. 0

B.  $\log_e 2$

C.  $\log_e \left( \frac{3}{2} \right)$

D.  $\log_e \left( \frac{2}{3} \right)$

Answer: B

Solution:

$$\lim_{n \rightarrow \infty} \left( \frac{1}{1+n} + \dots + \frac{1}{n+n} \right) = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n+r}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \left( \frac{1}{1 + \frac{r}{n}} \right)$$

$$= \int_0^1 \frac{1}{1+x} dx = [\ln(1+x)]_0^1 = \ln 2.$$

## Question45

$$\int_0^1 (x^{21} + x^{14} + x^7)(2x^{14} + 3x^7 + 6)^{1/7} dx = \frac{1}{l}(11)^{m/n}$$

where  $l, m, n \in \mathbb{N}$ ,  $m$  and  $n$  are coprime then  $l + m + n$  is equal to \_\_\_\_\_.

[1-Feb-2023 Shift 1]

Answer: 63

Solution:

$$\int (x^{20} + x^{13} + x^6)(2x^{14} + 3x^7 + 6x^7)^{1/7} dx$$

$$2x^{21} + 3x^{14} + 6x^7 = t$$

$$42(x^{20} + x^{13} + x^6) dx = dt$$

$$\frac{1}{42} \int_0^{11} t^{1/7} dt = \left( \frac{t^{8/7}}{\frac{8}{7}} \times \frac{1}{42} \right)_0^{11}$$

$$= \frac{1}{48} \left( t^{8/7} \right)_0^{11} = \frac{1}{48} (11)^{8/7}$$

$$l = 48, m = 8, n = 7$$

$$l + m + n = 63$$


---

## Question46

The value of the integral  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx$  is:

[1-Feb-2023 Shift 2]

Options:

- A.  $\frac{\pi^2}{6}$
- B.  $\frac{\pi^2}{12\sqrt{3}}$
- C.  $\frac{\pi^2}{3\sqrt{3}}$
- D.  $\frac{\pi^2}{6\sqrt{3}}$

Answer: D

Solution:

Solution:

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx \dots (1)$$

$$x \rightarrow -x$$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{-x + \frac{\pi}{4}}{2 - \cos 2x} dx \dots (2)$$

$$(1) + (2)$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\frac{\pi}{2}}{2 - \cos 2x} dx$$

$$I = \frac{\pi}{4} \cdot 2 \int_0^{\frac{\pi}{4}} \frac{dx}{2 - \cos 2x}$$

$$I = \frac{\pi}{4} \cdot 2 \int_0^{\frac{\pi}{4}} \frac{(1 + \tan^2 x) dx}{2(1 + \tan^2 x) - (1 - \tan^2 x)}$$

$$I = \frac{\pi}{4} \int_0^{\frac{\pi}{4}} \frac{dt}{3t^2 + 1}$$

$$\Rightarrow I = \frac{\pi}{2\sqrt{3}} \tan^{-1} \sqrt{3}$$

$$I = \frac{\pi^2}{6\sqrt{3}}$$


---

## Question47

If  $\int_0^{\frac{\pi}{2}} \frac{5^{\cos x}(1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x) dx}{1 + 5^{\cos x}} = \frac{k\pi}{16}$ , then k is equal to \_\_\_\_\_.

[1-Feb-2023 Shift 2]



Answer: 13

Solution:

$$I = \int_0^{\pi} \frac{5^{\cos x}(1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x)}{1 + 5^{\cos x}} dx$$
$$I = \int_0^{\pi} \frac{5^{-\cos x}(1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x)}{1 + 5^{\cos x}} dx$$
$$2I = \int_0^{\pi} (1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x) dx$$
$$\text{not } I = \text{not } 2 \int_0^{\frac{\pi}{2}} (1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x) dx$$
$$I = \int_0^{\frac{\pi}{2}} (1 + \sin x(-\sin 3x) + \sin^2 x - \sin^3 x \sin 3x) dx$$
$$2I = \int_0^{\frac{\pi}{2}} (3 + \cos 4x + \cos^3 x \cos 3x - \sin^3 x \sin 3x) dx$$
$$2I = \int_0^{\frac{\pi}{2}} 3 + \cos 4x + \left( \frac{\cos 3x + 3 \cos x}{4} \right) \cos 3x - \sin 3x \left( \frac{3 \sin x - \sin 3x}{4} \right) dx$$
$$2I = \int_0^{\frac{\pi}{2}} \left( 3 + \cos 4x + \frac{1}{4} + \frac{3}{4} \cos 4x \right) dx$$
$$2I = \frac{13}{4} \times \frac{\pi}{2} + \frac{7}{4} \left( \frac{\sin 4x}{4} \right)_0^{\frac{\pi}{2}} \Rightarrow I = \frac{13\pi}{16}$$

## Question 48

Let  $f(x)$  be a function satisfying  $f(x) + f(\pi - x) = \pi^2$ ,  $\forall x \in \mathbb{R}$ . Then  $\int_0^{\pi} f(x) \sin x dx$  is equal to :  
[6-Apr-2023 shift 2]

Options:

A.  $\frac{\pi^2}{2}$

B.  $\pi^2$

C.  $2\pi^2$

D.  $\frac{\pi^2}{4}$

Answer: B

Solution:

$$I = \int_0^{\pi} f(x) \sin x dx \dots (1)$$

Apply king property

$$I = \int_0^{\pi} f(\pi - x) \sin(\pi - x) dx \dots (1)$$

Add

$$2I = \int_0^{\pi} f(x) + f(\pi - x) \sin x dx$$

$$2I = \int_0^{\pi} \pi^2 \sin x \, dx$$

$$2I = \pi^2 (\text{not})$$

$$I = \pi^2$$

Ans. Option 2

## Question 49

Let  $f(x) = \frac{x}{(1+x^n)^{\frac{1}{n}}}$ ,  $x \in \mathbb{R} - \{-1\}$ ,  $n \in \mathbb{N}$ ,  $n > 2$ . If  $f^n(x) = n$  (f of of ..... upto n times) (x), then  $\lim_{n \rightarrow \infty} \int_0^1 x^{n-2} (f^n(x)) \, dx$  is equal to \_\_\_\_\_.  
[6-Apr-2023 shift 2]

Answer: 0

Solution:

$$\text{Let } f(x) = \frac{x}{(1+x^n)^{\frac{1}{n}}}, x \in \mathbb{R} - \{-1\}, n \in \mathbb{N}, n > 2.$$

If  $f^n(x) = n$  (f of of ..... upto n times) (x)

$$\text{then } \lim_{n \rightarrow \infty} \int_0^1 x^{n-2} (f^n(x)) \, dx$$

$$f(f(x)) = \frac{x}{(1+2x^n)^{1/n}}$$

$$f(f(f(x))) = \frac{x}{(1+3x^n)^{1/n}}$$

$$\text{Similarly } f^n(x) = \frac{x}{(1+n \cdot x^n)^{1/n}}$$

$$\text{Now } \lim_{n \rightarrow \infty} \int \frac{x^{n-2} \cdot x \, dx}{(1+n \cdot x^n)^{1/n}} = \lim_{n \rightarrow \infty} \int \frac{x^{n-1} \cdot dx}{(1+n \cdot x^n)^{1/n}}$$

Now  $1 + nx^n = t$

$$n^2 \cdot x^{n-1} \, dx = dt$$

$$x^{n-1} \, dx = \frac{dt}{n^2}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^2} \int_1^{1+n} \frac{dt}{t^{1/n}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^2} \left[ \frac{t^{1-\frac{1}{n}}}{1-\frac{1}{n}} \right]_1^{1+n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n(n-1)} \left( (1+n)^{\frac{n-1}{n}} - 1 \right) \text{ Now let } n = \frac{1}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{\left(1 + \frac{1}{h}\right)^{1-h} - 1}{\frac{1}{h} \frac{(1-h)}{h}}$$

Using series expansion

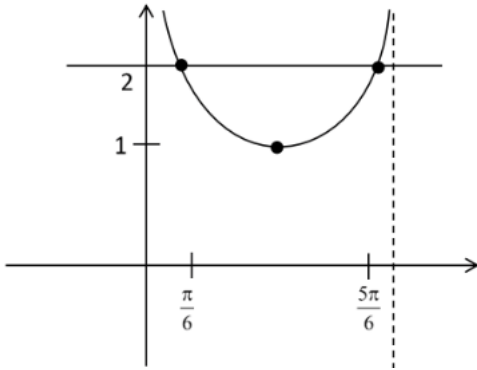
$\Rightarrow 0$

## Question 50

Let  $[t]$  denote the greatest integer  $\leq t$ . The  $\int_{\pi/6}^{5\pi/6} (8[\operatorname{cosec} x] - 5[\cot x]) \, dx$  is equal to \_\_\_\_\_  
[8-Apr-2023 shift 1]

Answer: 14

Solution:



$$8 \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [\operatorname{cosec} x] dx$$

$$8 \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} dx = \frac{16\pi/3}{16\pi/3}$$

$$I = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [\cot x] dx$$

$$x \rightarrow \pi - x$$

$$I = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [-\cot x] dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} ([\cot x] + [-\cot x]) dx$$

$$I = -\frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} dx \Rightarrow -\frac{1}{2} \left( \frac{4\pi}{6} \right)$$

$$= -\pi/3$$
$$\therefore \frac{2}{\pi} \left[ \frac{16\pi}{3} + \frac{5\pi}{3} \right] = \frac{2}{\pi} \left( \frac{21\pi}{3} \right)$$
$$= 14$$

## Question51

Let  $[t]$  denote the greatest integer function. If  $\int_0^{2.4} [x^2] dx = \alpha + \beta\sqrt{2} + \gamma\sqrt{3} + \delta\sqrt{5}$ , then  $\alpha + \beta + \gamma + \delta$  is equal to

[8-Apr-2023 shift 2]

Answer: 6

Solution:

$$\int_0^1 0 dx + \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 dx + \int_{\sqrt{3}}^2 3 dx + \int_2^{\sqrt{5}} 4 dx + \int_{\sqrt{5}}^{2.4} 5 dx$$

$$= \sqrt{2} - 1 + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) + 4(\sqrt{5} - 2) + 5((2 \cdot 4) - \sqrt{5})$$

$$= 9 - \sqrt{2} - \sqrt{3} - \sqrt{5}$$

$$\alpha + \beta + \gamma + \delta = 9 - 1 - 1 - 1 = 6$$


---

## Question52

Let  $f$  be a continuous function satisfying  $\int_0^t (f(x) + x^2) dx = \frac{4}{3}t^3, \forall t > 0$ . Then  $f\left(\frac{\pi^2}{4}\right)$  is equal to  
**[10-Apr-2023 shift 2]**

Options:

A.  $-\pi^2\left(1 + \frac{\pi^2}{16}\right)$

B.  $\pi\left(1 - \frac{\pi^3}{16}\right)$

C.  $-\pi\left(1 + \frac{\pi^3}{16}\right)$

D.  $\pi^2\left(1 - \frac{\pi^3}{16}\right)$

Answer: B

Solution:

$$\int_0^t (f(x) + x^2) dx = \frac{4}{3}t^3, \forall t > 0$$

$$(f(t^2) + t^4) = 2t$$

$$f(t^2) = 2t - t^4$$

$$t = \frac{\pi}{2} \Rightarrow f\left(\frac{\pi^2}{4}\right) = \frac{2\pi}{2} - \frac{\pi^4}{16}$$

$$= \pi - \frac{\pi^4}{16} = \pi\left(1 - \frac{\pi^3}{16}\right)$$


---

## Question53

The value of the integral  $\int_{-\log_e 2}^{\log_e 2} e^x (\log_e (e^x + \sqrt{1 + e^{2x}})) dx$  is equal to :  
**[11-Apr-2023 shift 1]**

Options:

A.  $\log_e \left( \frac{(2 + \sqrt{5})^2}{\sqrt{1 + \sqrt{5}}} \right) + \frac{\sqrt{5}}{2}$

B.  $\log_e \left( \frac{2(2 + \sqrt{5})^2}{\sqrt{1 + \sqrt{5}}} \right) - \frac{\sqrt{5}}{2}$

C.  $\log_e \left( \frac{\sqrt{2}(3 - \sqrt{5})^2}{\sqrt{1 + \sqrt{5}}} \right) + \frac{\sqrt{5}}{2}$

$$D. \log_e \left( \frac{\sqrt{2}(2+\sqrt{5})^2}{\sqrt{1+\sqrt{5}}} \right) - \frac{\sqrt{5}}{2}$$

**Answer: D**

**Solution:**

**Solution:**

$$I = \int_{-\ln 2}^{\ln 2} e^x (\ln(e^x + \sqrt{1+e^{2x}})) dx$$

$$\text{Put } e^x = t \Rightarrow e^x dx = dt$$

$$I = \int_{1/2}^2 \ln(t + \sqrt{1+t^2}) dt$$

Applying integration by parts.

$$= [t \ln(t + \sqrt{1+t^2})]_{1/2}^2 - \int_{1/2}^2 \frac{t}{t + \sqrt{1+t^2}} \left(1 + \frac{2t}{2\sqrt{1+t^2}}\right) dt$$

$$= 2 \ln(2 + \sqrt{5}) - \frac{1}{2} \ln\left(\frac{1+\sqrt{5}}{2}\right) - \int_{1/2}^2 \frac{t}{\sqrt{1+t^2}} dt$$

$$= 2 \ln(2 + \sqrt{5}) - \frac{1}{2} \ln\left(\frac{1+\sqrt{5}}{2}\right) - \frac{\sqrt{5}}{2}$$

$$= \ln \left( \frac{(2 + \sqrt{5})^2}{\left(\left(\frac{\sqrt{5}+1}{2}\right)^{\frac{1}{2}}\right)^2} \right) - \frac{\sqrt{5}}{2}$$

## Question54

For  $m, n > 0$ , let  $\alpha(m, n) = \int_0^2 t^m (1+3t)^n dt$ . If  $11\alpha(10, 6) + 18\alpha(11, 5) = p(14)^6$ , then  $p$  is equal to \_\_\_\_\_.  
[11-Apr-2023 shift 1]

**Answer: 32**

**Solution:**

**Solution:**

$$\alpha(m, n) = \int_0^2 t^m (1+3t)^n dt$$

$$\text{If } 11\alpha(10, 6) + 18\alpha(11, 5) = p(14)^6 \text{ then } P$$

$$= 11 \int_0^2 \frac{t^{10}}{11} \frac{(1+3t)^6}{1} + 10 \int_0^2 t^{11} (1+3t)^5 dt$$

$$= 11 \left[ (1+3t)^6 \cdot \frac{t^{11}}{11} - \int 6(1+3t)^5 \cdot 3 \frac{t^{11}}{11} \right]_0^2 + 18 \int_0^2 t^{11} (1+3t)^5 dt$$

$$= (t^{11}(1+3t)^6)_0^2$$

$$= 2^{11}(7)^6$$

$$= 2^5(14)^6$$

$$= 32(14)^6$$

## Question55

Let the function  $f : [0, 2] \rightarrow \mathbb{R}$  be defined as

$$f(x) = \begin{cases} e^{\min\{x^2, x-[x]\}} & x \in [0, 1) \\ e^{[x] - \log_e x} & x \in [1, 2) \end{cases}$$



where  $[t]$  denotes the greatest integer less than or equal to  $t$ . Then the value of the integral  $\int_0^2 xf(x) dx$  is  
**[11-Apr-2023 shift 2]**

**Options:**

A.  $(e-1)\left(e^2 + \frac{1}{2}\right)$

B.  $1 + \frac{3e}{2}$

C.  $2e - \frac{1}{2}$

D.  $2e - 1$

**Answer: A**

**Solution:**

$$F: [0, 2] \rightarrow \mathbb{R}$$

$$F(x) = \begin{cases} \min\{x^2, \{x\}\}; & x \in [0, 1) \\ [x - \log_e x] = 1; & x \in [1, 2) \end{cases}$$

$$F(x) = \begin{cases} e^{x^2}; & x \in [0, 1) \\ e & x \in [1, 2) \end{cases}$$

$$\int_0^2 xf(x) dx = \int_0^1 x \cdot e^{x^2} dx + \int_1^2 x \cdot e dx$$

$$= \frac{1}{2}(e-1) + \frac{1}{2}(4-1)e$$

$$\Rightarrow 2e - \frac{1}{2}$$

## Question 56

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying  $\int_0^{\frac{\pi}{2}} f(\sin 2x) \sin x dx + \alpha \int_0^{\frac{\pi}{4}} f(\cos 2x) \cos x dx = 0$ , then the value of  $\alpha$  is  
**[11-Apr-2023 shift 2]**

**Options:**

A.  $-\sqrt{3}$

B.  $\sqrt{3}$

C.  $-\sqrt{2}$

D.  $\sqrt{2}$

**Answer: C**

**Solution:**

$$F: \mathbb{R} \rightarrow \mathbb{R}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} F(\sin 2x) \sin x dx + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x) \cdot \cos x dx = 0$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} F(\sin 2x) \sin x \, dx + \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} F(\sin 2x) \cdot \sin x \, dx + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x) \cdot \cos x \, dx = 0$$

$$\int_0^a F(x) \, dx = \int_0^a F(a-x) \, dx$$

$$\text{Let } x = t + \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x) \sin \left( \frac{\pi}{4} - x \right) \, dx + \int_0^{\frac{\pi}{4}} F(\cos 2t) \sin \left( t + \frac{\pi}{4} \right) + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x) \cos x \, dx = 0$$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x) \left\{ \sin \left( \frac{\pi}{4} - x \right) + \sin \left( x + \frac{\pi}{4} \right) + \alpha \cos x \right\} \, dx = 0.$$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x) \{(\sqrt{2} + \alpha) \cos x\} \, dx = 0$$

$$(\sqrt{2} + \alpha) \int_0^{\frac{\pi}{4}} F(\cos 2x) \cos x \, dx = 0$$

$\therefore$  in interval  $\left(0, \frac{\pi}{4}\right) \Rightarrow F(\cos 2x) \& \cos x$  is NOT Zero.

$$\therefore \sqrt{2} + \alpha = 0$$

$$\alpha = -\sqrt{2}$$

## Question57

If  $\int_{-0.15}^{0.15} |100x^2 - 1| \, dx = \frac{k}{3000}$ , then k is equal to \_\_\_\_\_.  
[12-Apr-2023 shift 1]

**Answer: 575**

**Solution:**

$$\int_{-0.15}^{0.15} |100x^2 - 1| \, dx = 2 \int_0^{0.15} |100x^2 - 1| \, dx$$

$$\text{Now } 100x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{100} \Rightarrow x = 0.1$$

$$I = 2 \left[ \int_0^{0.1} (1 - 100x^2) \, dx + \int_{0.1}^{0.15} (100x^2 - 1) \, dx \right]$$

$$I = 2 \left[ x - \frac{100}{3}x^3 \right]_0^{0.1} + 2 \left[ \frac{100x^3}{3} - x \right]_{0.1}^{0.15}$$

$$= 2 \left[ 0.1 - \frac{0.1}{3} \right] + 2 \left[ \frac{0.3375}{3} - 0.15 - \frac{0.1}{3} + 0.1 \right]$$

$$= 2 \left[ 0.2 - \frac{0.2}{3} + 0.1125 - 0.15 \right]$$

$$= 2 \left[ \frac{5}{100} - \frac{2}{30} + \frac{1125}{10000} \right] = 2 \left( \frac{1500 - 2000 + 3375}{30000} \right)$$

$$= \frac{575}{3000} \Rightarrow k = 575$$

## Question58

The value of  $\frac{e^{-\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} e^{-x} \tan^{50} x \, dx}{\int_0^{\frac{\pi}{4}} e^{-x} (\tan^{49} x + \tan^{51} x) \, dx}$  is

[13-Apr-2023 shift 2]

Options:

- A. 25
- B. 51
- C. 50
- D. 49

Answer: C

Solution:

$$\begin{aligned} \text{let } I_1 &= e^{-\pi/4} + \int_0^{\pi/4} e^{-x} \tan^{50} x \, dx \\ I_2 &= \int_0^{\pi/4} e^{-x} (\tan^{49} x + \tan^{51} x) \, dx \\ &= \int_0^{\pi/4} e^{-x} \tan^{49} x (\sec^2 x) \, dx \\ &= \left[ e^{-x} \frac{\tan^{50} x}{50} \right]_0^{\pi/4} + \frac{1}{50} \int_0^{\pi/4} e^{-x} \tan^{50} x \, dx \\ &= \frac{e^{-\pi/4}}{50} + \frac{1}{50} \int_0^{\pi/4} e^{-x} \tan^{50} x \, dx = \frac{I_1}{50} \\ \text{then } \frac{I_1}{I_2} &= 50 \end{aligned}$$

## Question 59

Let  $f_n = \int_0^{\pi/2} \left( \sum_{k=1}^n \sin^{k-1} x \right) \left( \sum_{k=1}^n (2k-1) \sin^{k-1} x \right) \cos x \, dx$ ,  $n \in \mathbb{N}$ . Then  $f_{21} - f_{20}$  is equal to \_\_\_\_\_

[13-Apr-2023 shift 2]

Answer: 41

Solution:

$$\begin{aligned} f_n &= \int_0^{\pi/2} \left( \sum_{k=1}^n \sin^{k-1} x \right) \left( \sum_{k=1}^n (2k-1) \sin^{k-1} x \right) \cos x \, dx \\ \sin x &= t \\ \cos x \, dx &= dt \\ f_n &= \int_0^1 \left( \sum_{k=1}^{n-1} t^k \right) \left( \sum_{k=1}^n (2k-1) t^{k-1} \right) dt \\ &= \int_0^1 (1+t+t^2+\dots+t^{n-1})(1+3t+5t^2+\dots+(2n-1)t^{n-2}) dt \\ &= \int_0^1 (1+t+t^2+\dots+t^{n-1})(1+3t+5t^2+\dots+(2n-1)t^{n-1}) dt \\ &+ \int_0^1 (1+3t+5t^2+\dots+(2n-1)t^{n-1}) dt \end{aligned}$$



$$\begin{aligned}
& + \int_0^1 (1+t+t^2+\dots+t^{n-1})(2n+1)t^n dt \\
f_{n+1} - f_n &= \int_0^1 (1+3t+5t^2+\dots+(2n+1)t^n)t^n dt \\
& + \int_0^1 (1+t+t^2+\dots+t^{n+1})(2n+1)t^n dt \\
\text{put } n &= 20 \\
f_{21} - f_{20} &= \int_0^1 (1+3t+5t^2+\dots+41 \cdot t^{20})t^{20} dt + \int_0^1 (1+t+t^2+\dots+t^{19})(41 \cdot t^{20}) dt \\
&= \left( \frac{1}{21} + \frac{3}{22} + \frac{5}{23} + \dots + \frac{39}{40} + \frac{41}{41} \right) + \left( \frac{41}{21} + \frac{41}{22} + \frac{41}{40} \right) \\
&= \frac{1+41}{21} + \frac{3+41}{22} + \dots + \frac{39+41}{40} + 1 = 40 + 1 = 41
\end{aligned}$$

## Question60

If  $\int_0^1 \frac{1}{(5+2x-2x^2)(1+e^{(2-4x)})} dx = \frac{1}{\alpha} \log_e \left( \frac{\alpha+1}{\beta} \right)$ ,  $\alpha, \beta > 0$ , then  $\alpha^4 - \beta^4$  is equal to  
**[15-Apr-2023 shift 1]**

Options:

- A. 19
- B. -21
- C. 21
- D. 0

Answer: C

Solution:

Solution:

$$I = \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{(2-4x)})} \dots \text{(i)}$$

$$x \rightarrow 1-x$$

$$I = \int_0^1 \frac{e^{2-4x} dx}{(5+2x-2x^2)(1+e^{(2-4x)})} \dots \text{(ii)}$$

Add (i) and (ii)

$$2I \int_0^1 \frac{dx}{5+2x-2x^2} = \int_0^1 \frac{dx}{2 \left( \frac{11}{4} - \left(x - \frac{1}{2}\right)^2 \right)}$$

$$I = \frac{1}{\sqrt{11}} \ln \left( \frac{\sqrt{11}+1}{\sqrt{10}} \right)$$

$$\alpha = \sqrt{11}$$

$$\beta = \sqrt{10}$$

$$\alpha^4 - \beta^4 = 121 - 100 = 21$$

## Question61

Let  $I(x) = \int \frac{x^2(x \sec^2 x + \tan x)}{(x \tan x + 1)^2} dx$ . If  $I(0) = 0$ , then  $I\left(\frac{\pi}{4}\right)$  is equal to :  
**[6-Apr-2023 shift 1]**

Options:

A.  $\log_e \frac{(\pi+4)^2}{16} + \frac{\pi^2}{4(\pi+4)}$

B.  $\log_e \frac{(\pi+4)^2}{32} - \frac{\pi^2}{4(\pi+4)}$



C.  $\log_e \frac{(\pi+4)^2}{16} - \frac{\pi^2}{4(\pi+4)}$

D.  $\log_e \frac{(\pi+4)^2}{32} + \frac{\pi^2}{4(\pi+4)}$

**Answer: B**

**Solution:**

**Solution:**

$$I(x) = \int \frac{x^2(x \sec^2 x + \tan x)}{(x \tan x + 1)^2} dx$$

Let  $x \tan x + 1 = t$

$$I = x^2 \left( \frac{-1}{x \tan x + 1} \right) + \int \frac{2x}{x \tan x + 1} dx$$

$$I = x^2 \left( \frac{-1}{x \tan x + 1} \right) + 2 \int \frac{2x}{x \tan x + 1} dx$$

$$I = x^2 \left( \frac{-1}{x \tan x + 1} \right) + 2 \ln |x \sin x + \cos x| + C$$

As  $I(0) = 0 \Rightarrow C = 0$

$$I\left(\frac{\pi}{4}\right) = \ln\left(\frac{(\pi+4)^2}{32}\right) - \frac{\pi^2}{4(\pi+4)}$$

## Question62

Let  $I(x) = \int \frac{(x+1)}{x(1+xe^{xy})^2} dx$ ,  $x > 0$ . If  $\lim_{x \rightarrow \infty} I(x) = 0$ , then  $I(1)$  is equal to

**[8-Apr-2023 shift 1]**

**Options:**

A.  $\frac{e+1}{e+2} - \log_e(e+1)$

B.  $\frac{e+2}{e+1} + \log_e(e+1)$

C.  $\frac{e+2}{e+1} - \log_e(e+1)$

D.  $\frac{e+1}{e+2} + \log_e(e+1)$

**Answer: D**

## Question63

The integral  $\int \left( \left(\frac{x}{2}\right)^x + \left(\frac{2}{x}\right)^x \right) \log_2 x dx$  is equal to

**[8-Apr-2023 shift 2]**

**Options:**

A.  $\left(\frac{x}{2}\right)^x \log_2\left(\frac{2}{x}\right) + C$

B.  $\left(\frac{x}{2}\right)^x - \left(\frac{2}{x}\right)^x + C$

$$C. \left(\frac{x}{2}\right)^x \log_2\left(\frac{x}{2}\right) + C$$

$$D. \left(\frac{x}{2}\right)^x + \left(\frac{2}{x}\right)^x + C$$

**Answer: B**

**Solution:**

**Solution:**

$$\begin{aligned} & \int (x^x 2^{-x} + 2^x x^{-x}) \log_2^x dx \\ & \int (e^{x \ln x} \cdot e^{-x \ln 2} + e^{x \ln 2} \cdot e^{-x \ln x}) dx \\ & \int (e^{x \ln x - x \ln 2} + e^{x \ln 2 - x \ln x}) \frac{\ln x}{\ln 2} dx \\ & \text{let } x \ln x - x \ln 2 = t \\ & (\ln x + 1 - \ln 2) dx = dt \end{aligned}$$

## Question 64

If  $I(x) = \int e^{\sin^2 x} (\cos x \sin 2x - \sin x) dx$  and  $I(0) = 1$ , then  $I\left(\frac{\pi}{3}\right)$  is equal to :

[10-Apr-2023 shift 1]

**Options:**

A.  $e^{\frac{3}{4}}$

B.  $-e^{\frac{3}{4}}$

C.  $\frac{1}{2}e^{\frac{3}{4}}$

D.  $-\frac{1}{2}e^{\frac{3}{4}}$

**Answer: C**

**Solution:**

**Solution:**

$$\begin{aligned} I &= \int e^{\sin^2 x} (\sin 2x \cos x - \sin x) dx - \int e^{\sin^2 x} \sin x dx \\ &= \cos x \int e^{\sin^2 x} \sin 2x dx - \int ((-\sin x) \int e^{\sin^2 x} \sin 2x dx) dx - \int e^{\sin^2 x} \sin x dx \\ \sin^2 x &= t \\ \sin 2x dx &= dt \\ &= \cos x \int e^t dt + \int (\sin x \int e^t dt) dx - \int e^{\sin^2 x} \sin x dx \\ &= e^{\sin^2 x} \cos x + \int e^{\sin^2 x} \sin x dx - \int e^{\sin^2 x} \sin x dx \\ I &= e^{\sin^2 x} \cos x + C \\ I(0) &= 1 \\ \Rightarrow 1 &= 1 + C \\ \Rightarrow C &= 0 \\ \therefore I &= e^{\sin^2 x} \cos x \\ I\left(\frac{\pi}{3}\right) &= e^{\sin^2 \frac{\pi}{3}} \cos \frac{\pi}{3} \\ &= \frac{e^{\frac{3}{4}}}{2} \end{aligned}$$



## Question65

For  $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ , if  $\int \left( \left( \frac{x}{e} \right)^{2x} + \left( \frac{e}{x} \right)^{2x} \right) \log_e x dx = \frac{1}{\alpha} \left( \frac{x}{e} \right)^{\beta x} - \frac{1}{\gamma} \left( \frac{e}{x} \right)^{\delta x} + C$ , where  $e = \sum_{n=0}^{\infty} \frac{1}{n!}$  and  $C$  is constant of integration, then  $\alpha + 2\beta + 3\gamma - 4\delta$  is equal to  
[10-Apr-2023 shift 2]

Options:

- A. 4
- B. -4
- C. -8
- D. 1

Answer: A

Solution:

Solution:

$$\begin{aligned} x &= e^{\ln x} \\ \int \left( \left( \frac{x}{e} \right)^{2x} + \left( \frac{e}{x} \right)^{2x} \right) \log_e x dx &= \int [e^{2(x \ln x - x)} + e^{-2(x \ln x - x)}] \ln x dx \\ x \ln x - x &= t \\ \ln x \cdot dx &= dt \\ \int (e^{2t} + e^{-2t}) dt & \\ \frac{e^{2t}}{2} - \frac{e^{-2t}}{2} + C & \\ = \frac{1}{2} \left( \frac{x}{e} \right)^{2x} - \frac{1}{2} \left( \frac{e}{x} \right)^{2x} + C & \\ \alpha = \beta = \gamma = \delta = 2 & \\ \alpha + 2\beta + 3\gamma - 4\delta = 4 & \end{aligned}$$

## Question66

$\int_0^{\infty} \frac{6}{e^{3x} + 6e^{2x} + 11e^x + 6} dx =$   
[13-Apr-2023 shift 1]

Options:

- A.  $\log_e \left( \frac{32}{27} \right)$
- B.  $\log_e \left( \frac{256}{81} \right)$
- C.  $\log_e \left( \frac{512}{81} \right)$
- D.  $\log_e \left( \frac{64}{27} \right)$

Answer: A

Solution:

$$\begin{aligned}
I &= \int_0^{\infty} \frac{6}{(e^x+1)(e^x+2)(e^x+3)} dx \\
&= 6 \int_0^{\infty} \left( \frac{\frac{1}{2}}{e^x+1} + \frac{-1}{e^x+2} + \frac{\frac{1}{2}}{e^x+3} \right) dx \\
&= 3 \int_0^{\infty} \frac{e^{-x}}{1+e^{-x}} dx - 6 \int_0^{\infty} \frac{e^{-x}}{1+2e^{-x}} dx + 3 \int_0^{\infty} \frac{e^{-x}}{1+3e^{-x}} dx \\
&= 3[-\ln(1+e^{-x})]_0^{\infty} + 6 \frac{1}{2} [\ln(1+2e^{-x})]_0^{\infty} \\
&\quad - \frac{3}{3} [\ln(1+3e^{-x})]_0^{\infty} \\
&= 3 \ln 2 - 3 \ln 3 + \ln 4 \\
&= 3 \ln \frac{2}{3} + \ln 4 \\
&= \ln \frac{32}{27}
\end{aligned}$$

## Question 67

Let  $f(x) = \int \frac{dx}{(3+4x^2)\sqrt{4-3x^2}}$ ,  $|x| < \frac{2}{\sqrt{3}}$ . If  $f(0) = 0$  and  $f(1) = \frac{1}{\alpha\beta} \tan^{-1}\left(\frac{\alpha}{\beta}\right)$ ,  $\alpha, \beta > 0$ , then  $\alpha^2 + \beta^2$  is equal to

[15-Apr-2023 shift 1]

**Answer: 28**

**Solution:**

**Solution:**

$$\begin{aligned}
f(x) &= \int \frac{dx}{(3+4x^2)\sqrt{4-3x^2}} \\
x &= \frac{1}{t} \\
&= \int \frac{\frac{-1}{t^2} dt}{\frac{(3t^2+4)\sqrt{4t^2-3}}{t}} \\
&= \int \frac{-dt \cdot t}{(3t^2+4)\sqrt{4t^2-3}} : \text{ Put } 4t^2-3 = z^2 \\
&= -\frac{1}{4} \int \frac{z dx}{\left(3\left(\frac{z^2+3}{4}\right)+4\right)z} \\
&= \int \frac{-dz}{3z^2+25} = -\frac{1}{3} \int \frac{dz}{z^2 + \left(\frac{5}{\sqrt{3}}\right)^2} \\
&= -\frac{1}{3} \frac{\sqrt{3}}{5} \tan^{-1}\left(\frac{\sqrt{3}z}{5}\right) + C \\
&= -\frac{1}{5\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}}{5}\sqrt{4t^2-3}\right) + C \\
f(x) &= -\frac{1}{5\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}}{5}\sqrt{\frac{4-3x^2}{x^2}}\right) + C \\
\because f(0) = 0 \therefore c &= \frac{\pi}{10\sqrt{3}} \\
f(1) &= -\frac{1}{5\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}}{5}\right) + \frac{\pi}{10\sqrt{3}} \\
f(1) &= \frac{1}{5\sqrt{3}} \cot^{-1}\left(\frac{\sqrt{3}}{5}\right) = \frac{1}{5\sqrt{3}} \tan^{-1}\left(\frac{5}{\sqrt{3}}\right) \\
\alpha = 5 : \beta = \sqrt{3} \therefore \alpha^2 + \beta^2 &= 28
\end{aligned}$$



## Question68

The value of  $12 \int_0^3 |x^2 - 3x + 2| dx$  is  
[24-Jan-2023 Shift 1]

Answer: 22

Solution:

$$\begin{aligned} & 12 \int_0^3 |x^2 - 3x + 2| dx \\ &= 12 \int_0^3 \left| \left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \right| dx \\ \text{If } x - \frac{3}{2} &= t \\ dx &= dt \\ &= 24 \int_0^{3/2} \left| t^2 - \frac{1}{4} \right| dt \\ &= 24 \left[ -\int_0^{1/2} \left(t^2 - \frac{1}{4}\right) dt + \int_{1/2}^{3/2} \left(t^2 - \frac{1}{4}\right) dt \right] = 22 \end{aligned}$$

---

## Question69

If  $\int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx = g(x) + c$ ,  $g(1) = 0$ , then  $g\left(\frac{1}{2}\right)$  is equal to:

[26-Jun-2022-Shift-2]

Options:

- A.  $\log_e \left( \frac{\sqrt{3}-1}{\sqrt{3}+1} \right) + \frac{\pi}{3}$
- B.  $\log_e \left( \frac{\sqrt{3}+1}{\sqrt{3}-1} \right) + \frac{\pi}{3}$
- C.  $\log_e \left( \frac{\sqrt{3}+1}{\sqrt{3}-1} \right) - \frac{\pi}{3}$
- D.  $\frac{1}{2} \log_e \left( \frac{\sqrt{3}-1}{\sqrt{3}+1} \right) - \frac{\pi}{6}$

Answer: A

Solution:

$$\therefore \int_x^1 \sqrt{\frac{1-x}{1+x}} dx = g(x) + c$$

$$\int_1^{\frac{1}{2}} \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx = g\left(\frac{1}{2}\right) - g(1)$$

$$\therefore g\left(\frac{1}{2}\right) = \int_1^{\frac{1}{2}} \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx$$

$$\cot x = \cos 2\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\cos 2\theta} \cdot \frac{\sin \theta}{\cos \theta} (-2 \sin 2\theta) d\theta$$

$$= -\int_0^{\frac{\pi}{6}} \frac{4\sin^2 \theta}{\cos 2\theta} d\theta$$

$$= 2 \int_0^{\frac{\pi}{6}} \frac{(1 - 2\sin^2 \theta) - 1}{\cos 2\theta} d\theta$$

$$= 2 \int_0^{\frac{\pi}{6}} (1 - \sec 2\theta) d\theta$$

$$= \frac{\pi}{3} - 2 \cdot \frac{1}{2} [\ln |\sec 2\theta + \tan 2\theta|]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{3} - [\ln |2 + \sqrt{3}| - \ln 1]$$

$$= \frac{\pi}{3} + \ln \left( \frac{1}{2 - \sqrt{3}} \right)$$

$$= \frac{\pi}{3} + \ln \left| \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \right|$$

## Question 70

If  $\int \frac{(x^2+1)e^x}{(x+1)^2} dx = f(x)e^x + C$ , where  $C$  is a constant, then  $\frac{d^3f}{dx^3}$  at  $x = 1$  is equal to :

[27-Jun-2022-Shift-1]

Options:

A.  $-\frac{3}{4}$

B.  $\frac{3}{4}$

C.  $-\frac{3}{2}$

D.  $\frac{3}{2}$

Answer: B

Solution:

Solution:

$$I = \int \frac{e^x(x^2+1)}{(x+1)^2} dx = f(x)e^x + c$$

$$= \int \frac{e^x(x^2 - 1 + 1 + 1)}{(x+1)^2} dx$$

$$= \int e^x \left[ \frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx$$

$$= e^x \left( \frac{x-1}{x+1} \right) + c$$

$$\therefore f(x) = \frac{x-1}{x+1}$$

$$f(x) = 1 - \frac{2}{x+1}$$

$$f'(x) = 2 \left( \frac{1}{x+1} \right)^2$$

$$f''(x) = -4 \left( \frac{1}{x+1} \right)^3$$

$$f'''(x) = \frac{12}{(x+1)^4}$$

for  $x = 1$

$$f'''(1) = \frac{12}{2^4} = \frac{12}{16} = \frac{3}{4}$$

## Question 71

The value of the integral

$\int_{-\pi/2}^{\pi/2} \frac{dx}{(1+e^x)(\sin^6 x + \cos^6 x)}$  is equal to

[24-Jun-2022-Shift-2]

Options:

A.  $2\pi$

B. 0

C.  $\pi$

D.  $\frac{\pi}{2}$

Answer: C

Solution:

$$I = \int_{-\pi/2}^{\pi/2} \frac{dx}{(1+e^x)(\sin^6 x + \cos^6 x)} \dots (i)$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{dx}{(1+e^{-x})(\sin^6 x + \cos^6 x)} \dots (ii)$$

(i) and (ii)

From equation (i) & (ii)

$$2I = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sin^6 x + \cos^6 x}$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{dx}{\sin^6 x + \cos^6 x} = \int_0^{\pi/2} \frac{dx}{1 - \frac{3}{4} \sin^2 2x}$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{4 \sec^2 2x dx}{4 + \tan^2 2x} = 2 \int_0^{\pi/4} \frac{4 \sec^2 2x}{4 + \tan^2 2x} dx$$

when  $x = 0, t = 0$  Now,  $\tan 2x = t$  when  $x = \frac{\pi}{4}, t \rightarrow \infty$

$$2 \sec^2 2x dx = dt$$

$$\therefore I = 2 \int_0^{\infty} \frac{2 dt}{4 + t^2} = 2 \left( \tan^{-1} \frac{t}{2} \right)_0^{\infty}$$

$$= 2 \frac{\pi}{2} = \pi$$

## Question 72



$$\lim_{n \rightarrow \infty} \left( \frac{n^2}{(n^2+1)(n+1)} + \frac{n^2}{(n^2+4)(n+2)} + \frac{n^2}{(n^2+9)(n+3)} + \dots + \frac{n^2}{(n^2+n^2)(n+n)} \right)$$

is equal to :

[24-Jun-2022-Shift-2]

Options:

A.  $\frac{\pi}{8} + \frac{1}{4} \log_e 2$

B.  $\frac{\pi}{4} + \frac{1}{8} \log_e 2$

C.  $\frac{\pi}{4} - \frac{1}{8} \log_e 2$

D.  $\frac{\pi}{8} + \log_e \sqrt{2}$

Answer: A

Solution:

$$\lim_{n \rightarrow \infty} \left( \frac{n^2}{(n^2+1)(n+1)} + \frac{n^2}{(n^2+4)(n+2)} + \dots + \frac{n^2}{(n^2+n^2)(n+n)} \right)$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n^2}{(n^2+r^2)(n+r)}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \frac{1}{\left[1 + \left(\frac{r}{n}\right)^2\right] \left[1 + \left(\frac{r}{n}\right)\right]}$$

$$= \int_0^1 \frac{1}{(1+x^2)(1+x)} dx$$

$$= \frac{1}{2} \int_0^1 \left[ \frac{1}{1+x} - \frac{(x-1)}{(1+x^2)} \right] dx$$

$$= \frac{1}{2} \left[ \ln(1+x) - \frac{1}{2} \ln(1+x^2) + \tan^{-1} x \right]_0^1$$

$$= \frac{1}{2} \left[ \frac{\pi}{4} + \frac{1}{2} \ln 2 \right] = \frac{\pi}{8} + \frac{1}{4} \ln 2$$

## Question 73

The value of  $\int_0^\pi \frac{e^{\cos x} \sin x}{(1 + \cos^2 x)(e^{\cos x} + e^{-\cos x})} dx$  is equal to:

[25-Jun-2022-Shift-1]

Options:

A.  $\frac{\pi^2}{4}$

B.  $\frac{\pi^2}{2}$

C.  $\frac{\pi}{4}$

D.  $\frac{\pi}{2}$

Answer: C

Solution:

$$\int_0^{\pi} \frac{e^{\cos x} \sin x}{(1 + \cos^2 x)(e^{\cos x} - e^{-\cos x})} dx$$

Let  $\cos x = t$

$$\sin x dx = dt$$

$$= \int_1^{-1} \frac{-e^t dt}{(1+t^2)(e^t - e^{-t})}$$

$$I = \int_{-1}^1 \frac{e^t}{(1+t^2)(e^t - e^{-t})} dt \dots (i)$$

$$I = \int_{-1}^1 \frac{e^{-t}}{(1+t^2)(e^{-t} + e^t)} dt \dots (ii)$$

Adding (i) and (ii)

Adding (i) and (ii)

$$2I = \int_{-1}^1 \frac{dt}{1-t^2}$$

$$2I = \tan^{-1} \Big|_{-1}^1$$

$$2I = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

## Question 74

Let  $g : (0, \infty) \rightarrow \mathbf{R}$  be a differentiable function such that

$$\int \left( \frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x + 1)^2} \right) dx = \frac{xg(x)}{e^x + 1} + c$$

, for all  $x > 0$ , where  $c$  is an arbitrary constant. Then :

[25-Jun-2022-Shift-1]

Options:

A.  $g$  is decreasing in  $\left(0, \frac{\pi}{4}\right)$

B.  $g'$  is increasing in  $\left(0, \frac{\pi}{4}\right)$

C.  $g + g'$  is increasing in  $\left(0, \frac{\pi}{2}\right)$

D.  $g - g'$  is increasing in  $\left(0, \frac{\pi}{2}\right)$

**Answer: D**

**Solution:**

$$\int \left( \frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x - 1)^2} \right) dx = \frac{xg(x)}{e^x + 1} + c$$

On differentiating both sides w.r.t.  $x$ , we get

$$\left( \frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x + 1)^2} \right)'$$

$$= \frac{(e^x + 1)(g(x) + xg'(x)) - e^x \cdot x \cdot g(x)}{(e^x + 1)^2}$$

$$(e^x + 1)x(\cos x - \sin x) + g(x)(e^x - 1 - xe^x)$$

$$= (e^x + 1)(g(x) + xg'(x)) - e^x \cdot x \cdot g(x)$$

$$\Rightarrow g'(x) = \cos x - \sin x$$

$$\Rightarrow g(x) = \sin x - \cos x + C$$

$g(x)$  is increasing in  $(0, \pi/4)$

$$g''(x) = -\sin x - \cos x < 0$$

$$\Rightarrow g'(x) \text{ is decreasing function let } h(x) = g(x) + g'(x) = 2 \cos x + C \Rightarrow h'(x) = g'(x) + g''(x) = -2 \sin x < 0$$

$$\Rightarrow h \text{ is decreasing let } \phi(x) = g(x) - g'(x) = 2 \sin x + C \Rightarrow \phi'(x) = g'(x) - g''(x) = 2 \cos x > 0 \Rightarrow \phi \text{ is increasing}$$

## Question 75

If  $b_n = \int_0^{\frac{\pi}{2}} \frac{\cos^2 nx}{\sin x} dx$ ,  $n \in \mathbf{N}$ , then

[25-Jun-2022-Shift-2]

Options:

- A.  $b_3 - b_2, b_4 - b_3, b_5 - b_4$  are in A.P. with common difference  $-2$
- B.  $\frac{1}{b_3 - b_2}, \frac{1}{b_4 - b_3}, \frac{1}{b_5 - b_4}$  are in an A.P. with common difference  $2$
- C.  $b_3 - b_2, b_4 - b_3, b_5 - b_4$  are in a G.P.
- D.  $\frac{1}{b_3 - b_2}, \frac{1}{b_4 - b_3}, \frac{1}{b_5 - b_4}$  are in an A.P. with common difference  $-2$

Answer: D

Solution:

$$b_n - b_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\cos^2 nx - \cos^2(n-1)x}{\sin x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{-\sin(2n-1)x \cdot \sin x}{\sin x} dx$$

$$= \left. \frac{\cos(2n-1)x}{2n-1} \right|_0^{\pi/2} = -\frac{1}{2n-1}$$

So,  $b_3 - b_2, b_4 - b_3, b_5 - b_4$  are in H.P.

$$\Rightarrow \frac{1}{b_3 - b_2}, \frac{1}{b_4 - b_3}, \frac{1}{b_5 - b_4} \text{ are in A.P. with common difference } -2.$$

## Question 76

The value of  $b > 3$  for which

$$12 \int_3^b \frac{1}{(x^2-1)(x^2-4)} dx = \log_e \left( \frac{49}{40} \right)$$

, is equal to  
[25-Jun-2022-Shift-2]

Answer: 6

Solution:

$$I = \int \frac{1}{(x^2-1)(x^2-4)} dx = \frac{1}{3} \int \left( \frac{1}{x^2-4} - \frac{1}{x^2-1} \right) dx$$

$$= \frac{1}{3} \left( \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| - \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| \right) + C$$

$$12I = \ln \left| \frac{x-2}{x+2} \right| + 2 \ln \left| \frac{x-1}{x+1} \right| + C$$

$$12 \int_3^b \frac{dx}{(x^2-4)(x^2-1)}$$

$$= \ln \left( \frac{b-2}{b+2} \right) - 2 \ln \left( \frac{b-1}{b+1} \right) - \left( \ln \left( \frac{1}{5} \right) - 2 \ln \left( \frac{1}{2} \right) \right)$$

$$= \ln \left( \left( \frac{b-2}{b+2} \right) \cdot \frac{(b+1)^2}{(b-1)^2} \right) - \left( \ln \frac{4}{5} \right)$$

$$\text{So, } \frac{49}{40} = \frac{(b-2)}{(b+2)} \frac{(b+1)^2}{(b-1)^2} \cdot \frac{5}{4}$$

$$\Rightarrow b = 6$$

## Question 77

Let  $f(x) = \max\{|x+1|, |x+2|, \dots, |x+5|\}$ . Then  $\int_{-6}^0 f(x) dx$  is equal to \_\_\_\_  
[26-Jun-2022-Shift-1]

Answer: 21

Solution:

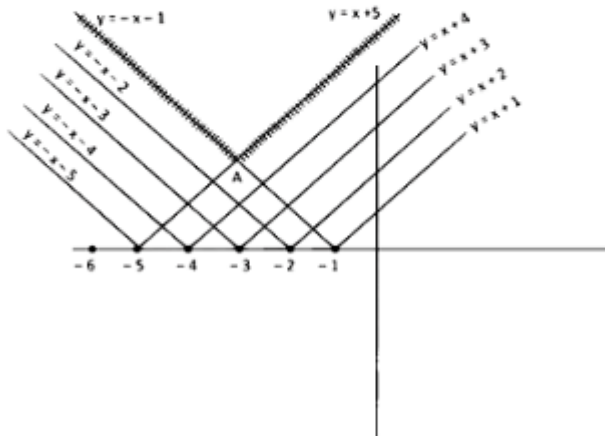
For  $|x+1|$  critical point,  $x+1=0 \Rightarrow x=-1$

For  $|x+2|$  critical point,  $x+2=0 \Rightarrow x=-2$

For  $|x+3|$  critical point,  $x+3=0 \Rightarrow x=-3$

For  $|x+4|$  critical point,  $x+4=0 \Rightarrow x=-4$

For  $|x+5|$  critical point,  $x+5=0 \Rightarrow x=-5$



Here maximum function is represent by the dotted line.

∴ Point of intersection  $A$  of line  $y = -x - 1$  and  $y = x + 5$  :

$$-x - 1 = x + 5$$

$$\Rightarrow 2x = -6$$

$$\Rightarrow x = -3$$

$$\therefore y = -3 + 5 = 2$$

∴ Point  $A = (-3, 2)$

$$\therefore \int_{-6}^0 f(x) dx$$

$$= \int_{-6}^{-3} (-x - 1) dx + \int_{-3}^0 (x + 5) dx$$

$$= \left( -\frac{x^2}{2} - x \right)_{-6}^{-3} + \left[ \frac{x^2}{2} + 5x \right]_{-3}^0$$

$$= \left[ \left( -\frac{9}{2} + 3 \right) - \left( -\frac{36}{2} + 6 \right) \right] + \left[ 0 - \left( \frac{9}{2} - 15 \right) \right]$$

$$= \left( -\frac{3}{2} + 12 \right) + \frac{21}{2}$$

$$= \frac{21}{2} = 21$$

---

## Question 78

The value of the integral

$$\frac{48}{\pi^4} \int_0^{\pi} \left( \frac{3\pi x^2}{2} - x^3 \right) \frac{\sin x}{1 + \cos^2 x} dx \text{ is equal to } \underline{\hspace{2cm}}$$

[26-Jun-2022-Shift-1]

Answer: 6

Solution:

$$I = \frac{48}{\pi^4} \int_0^{\pi} \left[ \left( \frac{\pi}{2} - x \right)^3 - \frac{3\pi^2}{4} \left( \frac{\pi}{2} - x \right) + \frac{\pi^3}{4} \right] \frac{\sin x dx}{1 + \cos^2 x}$$

$$\text{Using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$



$$I = \frac{48}{\pi^4} \int_0^\pi \left[ -\left(\frac{\pi}{2} - x\right)^3 + \frac{3\pi^4}{4} \left(\frac{\pi}{2} - x\right) + \frac{\pi^3}{4} \right] \frac{\sin x dx}{1 + \cos^2 x}$$

Adding these two equations, we get

$$2I = \frac{48}{\pi^4} \int_0^\pi \frac{\pi^3}{2} \cdot \frac{\sin x dx}{1 + \cos^2 x}$$

$$\Rightarrow I = \frac{12}{\pi} [-\tan^{-1}(\cos x)]_0^\pi = \frac{12}{\pi} \cdot \frac{\pi}{2} = 6$$

## Question 79

The integral  $\frac{24}{\pi} \int_0^{\sqrt{2}} \frac{(2-x^2)dx}{(2+x^2)\sqrt{4+x^4}}$  is equal to

[26-Jun-2022-Shift-2]

Answer: 6

Solution:

$$I = \frac{24}{\pi} \int_0^{\sqrt{2}} \frac{2-x^2}{(2+x^2)\sqrt{4+x^4}} dx$$

$$\text{Let } x = \sqrt{2}t \Rightarrow dx = \sqrt{2}dt$$

$$I = \frac{24}{\pi} \int_0^1 \frac{(2-2t^2) \cdot \sqrt{2}dt}{(2+2t^2)\sqrt{4+4t^4}}$$

$$= \frac{12\sqrt{2}}{\pi} \int_0^1 \frac{\left(\frac{1}{t^2}-1\right)dt}{\left(t+\frac{1}{t}\right)\sqrt{\left(t+\frac{1}{t}\right)^2-2}}$$

$$\text{Let } t + \frac{1}{t} = u$$

$$\Rightarrow \left(1 - \frac{1}{t^2}\right)dt = du$$

$$= \frac{12\sqrt{2}}{\pi} \int_2^\infty \frac{-du}{u\sqrt{4^2-2}}$$

$$= \frac{12\sqrt{2}}{\pi} \int_2^\infty \frac{du}{u^2 \sqrt{-\left(\frac{\sqrt{2}}{u}\right)^2}}$$

$$= \frac{12\sqrt{2}}{\pi} \int_0^1 \frac{-\frac{1}{\sqrt{2}}dp}{\frac{1}{\sqrt{2}}\sqrt{1-p^2}}$$

$$= \frac{12}{\pi} [\sin^{-1}p]_0^1$$

$$= \frac{12}{\pi} \cdot \frac{\pi}{4} = 3$$

## Question 80

The value of the integral

$\int_{-2}^2 \frac{x^3 + |x|}{(e^{|x|} + 1)} dx$  is equal to:

[27-Jun-2022-Shift-1]

Options:

- A.  $5e^2$
- B.  $3e^{-2}$
- C. 4
- D. 6

Answer: D

Solution:

$$I = \int_{-2}^2 \frac{|x^3 + x|}{e^{|x|} + 1} dx \dots (i)$$

$$I = \int_{-2}^2 \frac{|x^3 + x|}{e^{-|x|} + 1} dx \dots (ii)$$

$$2I = \int_{-2}^2 |x^3 + x| dx$$

$$2I = 2 \int_0^2 (x^3 + x) dx$$

$$I = \int_0^2 (x^3 + x) dx$$

$$= \left[ \frac{x^4}{4} + \frac{x^2}{2} \right]_0^2$$

$$= \left( \frac{16}{4} + \frac{4}{2} \right) - 0$$

$$= 4 + 2 = 6$$

## Question 81

Let  $f$  be a differentiable function in  $(0, \frac{\pi}{2})$ . If  $\int_{\cos x}^1 t^2 f(t) dt = \sin^3 x + \cos x$ , then  $\frac{1}{\sqrt{3}} f' \left( \frac{1}{\sqrt{3}} \right)$  is equal to

[27-Jun-2022-Shift-2]

Options:

- A.  $6 - 9\sqrt{2}$
- B.  $6 - \frac{9}{\sqrt{2}}$
- C.  $\frac{9}{2} - 6\sqrt{2}$
- D.  $\frac{9}{\sqrt{2}} - 6$

Answer: B

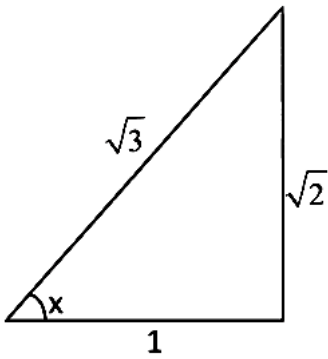
Solution:

$$\int_{\cos x}^1 t^2 f(t) dt = \sin^3 x + \cos x$$

$$\Rightarrow \sin x \cos^2 x f(\cos x) = 3 \sin^2 x \cos x - \sin x$$

$$\Rightarrow f(\cos x) = 3 \tan x - \sec^2 x$$

$$\Rightarrow f'(\cos x) \cdot (-\sin x) = 3 \sec^2 x - 2 \sec^2 x \tan x$$



Put  $\cos x = \frac{1}{\sqrt{3}}$ ,

$$\therefore f'\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{\sqrt{2}}{\sqrt{3}}\right) = 9 - 6\sqrt{2}$$

$$\frac{1}{\sqrt{3}}f'\left(\frac{1}{\sqrt{3}}\right) = 6 - \frac{9}{\sqrt{2}}$$

## Question 82

The integral  $\int_0^1 \frac{1}{7^{\lfloor \frac{1}{x} \rfloor}} dx$ , where  $\lfloor \cdot \rfloor$  denotes the greatest integer function, is equal to

[27-Jun-2022-Shift-2]

Options:

A.  $1 + 6\log_e\left(\frac{6}{7}\right)$

B.  $1 - 6\log_e\left(\frac{6}{7}\right)$

C.  $\log_e\left(\frac{7}{6}\right)$

D.  $1 - 7\log_e\left(\frac{6}{7}\right)$

Answer: A

Solution:

$$\int_0^1 \frac{1}{7^{\lfloor \frac{1}{x} \rfloor}} dx, \quad \text{let } \frac{1}{x} = t$$

$$\frac{-1}{x^2} dx = dt$$

$$= \int_{\infty}^1 \frac{1}{-t^2 7^{\lfloor t \rfloor}} dt = \int_1^{\infty} \frac{1}{t^2 7^{\lfloor t \rfloor}} dt$$

$$= \int_1^2 \frac{1}{7t^2} dt + \int_2^3 \frac{1}{7t^2} dt + \dots$$

$$= \frac{1}{7} \left[ -\frac{1}{t} \right]_1^2 + \frac{1}{7^2} \left[ -\frac{1}{t} \right]_2^3 + \frac{1}{7^3} \left[ -\frac{1}{t} \right]_3^4 + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{7^n} \left( \frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \sum_{n=1}^{\infty} \frac{\left(\frac{1}{7}\right)^n}{n} - 7 \sum_{n=1}^{\infty} \frac{\left(\frac{1}{7}\right)^{n+1}}{n+1}$$

$$= -\log\left(1 - \frac{1}{7}\right) + 7 \log\left(1 - \frac{1}{7}\right) + 1$$

$$= 1 + 6 \log \frac{6}{7}$$


---

## Question83

Let  $[t]$  denote the greatest integer less than or equal to  $t$ . Then, the value of the integral  $\int_0^1 [-8x^2 + 6x - 1] dx$  is equal to  
[28-Jun-2022-Shift-1]

Options:

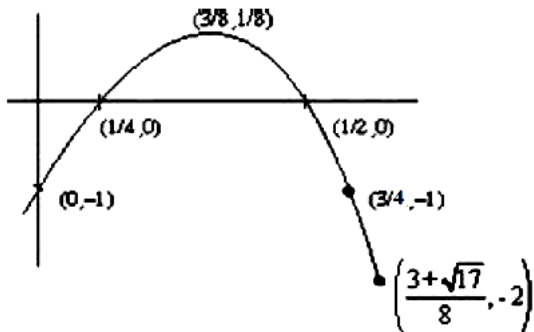
- A.  $-1$
- B.  $\frac{-5}{4}$
- C.  $\frac{\sqrt{17}-13}{8}$
- D.  $\frac{\sqrt{17}-16}{8}$

Answer: C

Solution:

$$\int_0^1 [-8x^2 + 6x - 1] dx$$

$$= \int_0^{1/4} -1 dx + \int_{1/4}^{3/4} 0 dx + \int_{3/4}^1 -1 dx$$



$$+ \int_{3/4}^{\frac{3+\sqrt{17}}{8}} -2 dx + \int_{\frac{3+\sqrt{17}}{8}}^1 \frac{1}{3+\sqrt{17}} dx$$

$$= -[x]_0^{1/4} + 0 - [x]_{1/2}^{3/4} - 2[x]_{3/4}^{\frac{3+\sqrt{17}}{8}} - 3[x]_{\frac{3+\sqrt{17}}{8}}^1$$

$$= -\left(\frac{1}{4} - 0\right) - \left(\frac{3}{4} - \frac{1}{2}\right) - 2\left(\frac{3+\sqrt{17}}{8} - \frac{3}{4}\right) - 3\left(1 - \frac{3+\sqrt{17}}{8}\right)$$

$$= -\frac{1}{4} - \frac{1}{4} + \frac{-6-2\sqrt{17}}{8} + \frac{3}{2} - 3 + \frac{9+3\sqrt{17}}{8}$$

$$= \frac{\sqrt{17}-13}{8}$$


---

## Question84

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function such that  $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ ,  $f\left(\frac{\pi}{2}\right) = 0$  and  $f'\left(\frac{\pi}{2}\right) = 1$  and

let  $g(x) = \int_x^{\pi/4} (f'(t) \sec t + \tan t \sec t f(t)) dt$  for  $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ . Then  $\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} g(x)$  is equal to :

[28-Jun-2022-Shift-2]

Options:

- A. 2
- B. 3
- C. 4
- D. -3

Answer: B

Solution:

Solution:

Given:  $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ ,  $f\left(\frac{\pi}{2}\right) = 0$  and  $f'\left(\frac{\pi}{2}\right) = 1$

$$g(x) = \int_x^{\pi/4} (f'(t) \sec t + \tan t \sec t f(t)) dt$$

$$= [\sec t + f(t)]_x^{\pi/4} = 2 - \sec x f(x)$$

$$\text{Now, } \lim_{x \rightarrow \frac{\pi}{2}} g(x) = \lim_{h \rightarrow 0} g\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h \rightarrow 0} 2 - (\operatorname{cosec} h) f\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h \rightarrow 0} \left[ 2 - \frac{f\left(\frac{\pi}{2} - h\right)}{\sin h} \right]$$

$$= \lim_{h \rightarrow 0} \left[ 2 + \frac{f'\left(\frac{\pi}{2} - h\right)}{\cos h} \right]$$

$$= 3$$

## Question 85

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying  $f(x) + f(x+k) = n$ , for all  $x \in \mathbb{R}$  where  $k > 0$  and  $n$  is a positive integer. If  $I_1 = \int_0^{4nk} f(x) dx$  and  $I_2 = \int_{-k}^{3k} f(x) dx$ , then :

[28-Jun-2022-Shift-2]

Options:

- A.  $I_1 + 2I_2 = 4nk$
- B.  $I_1 + 2I_2 = 2nk$
- C.  $I_1 + nI_2 = 4n^2k$
- D.  $I_1 + nI_2 = 6n^2k$

Answer: C



## Solution:

### Solution:

$$f : \mathbb{R} \rightarrow \mathbb{R} \text{ and } f(x) + f(x+k) = n \quad \forall x \in \mathbb{R}$$

$$x \rightarrow x+k$$

$$f(x+k) + f(x+2k) = n$$

$$\therefore f(x+2k) = f(x)$$

So, period of  $f(x)$  is  $2k$

$$\text{Now, } I_1 = \int_0^{4nk} f(x) dx = 2n \int_0^{2k} f(x) dx$$

$$= 2n \left[ \int_0^k f(x) dx + \int_k^{2k} f(x) dx \right]$$

$$x = t+k \Rightarrow dx = dt \text{ (in second integral)}$$

$$= 2n \left[ \int_0^k f(x) dx + \int_0^k f(t+k) dt \right]$$

$$= 2n^2 k$$

$$\text{Now, } I_2 = \int_{-k}^{3k} f(x) dx = 2 \int_0^{2k} f(x) dx$$

$$I_2 = 2(nk)$$

$$\therefore I_1 + nI_2 = 4n^2 k$$

## Question 86

$$\int_0^5 \cos \left( \pi \left( x - \left[ \frac{x}{2} \right] \right) \right) dx$$

where  $[t]$  denotes greatest integer less than or equal to  $t$ , is equal to:

[29-Jun-2022-Shift-1]

### Options:

A. -3

B. -2

C. 2

D. 0

Answer: D

### Solution:

We know,

$\left[ \frac{x}{2} \right]$  is discontinuous at 1, 2, 3, 4,.....

$\therefore [x]$  is discontinuous at 2, 4, 6, 8,.....

In between 0 to 5 it is discontinuous at 2 and 4.

Break the integration into 3 parts

(1) 0 to 2

(2) 2 to 4

(3) 4 to 5

$$\therefore \int_0^5 \cos \left( \pi \left( x - \left[ \frac{x}{2} \right] \right) \right) dx$$

$$= \int_0^2 \cos(\pi(x-0)) dx + \int_2^4 \cos(\pi(x-1)) dx + \int_4^5 \cos(\pi(x-2)) dx$$

$$= \int_0^2 \cos \pi x dx + \int_2^4 \cos(\pi x - \pi) dx + \int_4^5 \cos(\pi x - 2\pi) dx$$

$$= \int_0^2 \cos \pi x dx - \int_2^4 \cos \pi x dx + \int_4^5 \cos \pi x dx$$

$$= \left[ \frac{\sin \pi x}{\pi} \right]_0^2 - \left[ \frac{\sin \pi x}{\pi} \right]_2^4 + \left[ \frac{\sin \pi x}{\pi} \right]_4^5$$

$$= 0 - 0 + 0$$

## Question87

Let  $f$  be a real valued continuous function on  $[0, 1]$  and  $f(x) = x + \int_0^1 (x-t)f(t)dt$

Then, which of the following points  $(x, y)$  lies on the curve  $y = f(x)$  ?

[29-Jun-2022-Shift-2]

Options:

- A. (2, 4)
- B. (1, 2)
- C. (4, 17)
- D. (6, 8)

Answer: D

Solution:

$$f(x) = \left( 1 + \int_0^1 f(t) dt \right) x - \int_0^1 tf(t) dt$$

$$f(x) = Ax - B$$

$$A = 1 + \int_0^1 f(t) dt = 1 + \int_0^1 (At - B) dt$$

$$\Rightarrow A = 2(1 - B)$$

$$\text{Also } B = \int_0^1 tf(t) dt = \int_0^1 (At^2 - Bt) dt$$

$$A = \frac{9}{2}B$$

From (2), (3)

$$A = \frac{18}{13}, B = \frac{4}{13}$$

$$\text{so } f(6) = 8$$

---

## Question88

If  $\int_0^2 (\sqrt{2x} - \sqrt{2x-x^2}) dx = \int_0^1 \left( 1 - \sqrt{1-y^2} - \frac{y^2}{2} \right) dy + \int_1^2 \left( 2 - \frac{y^2}{2} \right) dy + I$ , then  $I$  equals

[29-Jun-2022-Shift-2]

Options:

- A.  $\int_0^1 (1 + \sqrt{1-y^2}) dy$
- B.  $\int_0^1 \left( \frac{y^2}{2} - \sqrt{1-y^2} + 1 \right) dy$
- C.  $\int_0^1 (1 - \sqrt{1-y^2}) dy$
- D.  $\int_0^1 \left( \frac{y^2}{2} + \sqrt{1-y^2} + 1 \right) dy$

Answer: C

Solution:



$$\text{LHS} = \int_0^2 (\sqrt{2x} - \sqrt{2x-x^2}) dx = \frac{8}{3} - \frac{\pi}{2}$$

$$\text{RHS} = \int_0^1 \left(1 - \sqrt{1-y^2} - \frac{y^2}{2}\right) dy + \int_1^2 \left(2 - \frac{y^2}{2}\right) dy + I$$

$$I + \frac{5}{3} - \frac{\pi}{4}$$

$$\text{So, } I = 1 - \frac{\pi}{4} = \int_0^1 (1 - \sqrt{1-y^2}) dy$$

## Question89

Let  $f(\theta) = \sin \theta + \int_{-\pi/2}^{\pi/2} (\sin \theta + t \cos \theta) f(t) dt$ . Then the value of  $\left| \int_0^{\pi/2} f(\theta) d\theta \right|$  is

[24-Jun-2022-Shift-1]

Answer: 1

Solution:

$$f(\theta) = \sin \theta \left(1 + \int_{-\pi/2}^{\pi/2} f(t) dt\right) + \cos \theta \left(\int_{-\pi/2}^{\pi/2} t f(t) dt\right)$$

$$\text{Clearly } f(\theta) = a \sin \theta + b \cos \theta$$

$$\text{Where } a = 1 + \int_{-\pi/2}^{\pi/2} (a \sin t + b \cos t) dt \Rightarrow a = 1 + 2b \dots (i) \text{ and } b = \int_{-\pi/2}^{\pi/2} (at \sin t + bt \cos t) dt \Rightarrow b = 2a \dots (ii) \text{ from (i) and (ii) we get}$$

$$a = -\frac{1}{3} \text{ and } b = -\frac{2}{3}$$

$$\text{So } f(\theta) = -\frac{1}{3}(\sin \theta + 2 \cos \theta)$$

$$\Rightarrow \left| \int_0^{\pi/2} f(\theta) d\theta \right| = \frac{1}{3}(1 + 2 \times 1) = 1$$

## Question90

$$\text{Let } \text{Max}_{0 \leq x \leq 2} \left\{ \frac{9-x^2}{5-x} \right\} = \alpha \text{ and } \text{Min}_{0 \leq x \leq 2} \left\{ \frac{9-x^2}{5-x} \right\} = \beta.$$

$$\text{If } \int_{\beta - \frac{8}{3}}^{2\alpha - 1} \text{Max} \left\{ \frac{9-x^2}{5-x}, x \right\} dx = \alpha_1 + \alpha_2 \log_e \left( \frac{8}{15} \right) \text{ then } \alpha_1 + \alpha_2 \text{ is equal to}$$

[24-Jun-2022-Shift-1]

Answer: 34

Solution:



$$\text{Let } f(x) = \frac{x^2-9}{x-5} \Rightarrow f'(x) = \frac{(x-1)(x-9)}{(x-5)^2}$$

$$\text{So, } \alpha = f(1) = 2 \text{ and } \beta = \min(f(0), f(2)) = \frac{5}{3}$$

$$\begin{aligned} \text{Now, } \int_{-1}^3 \max \left\{ \frac{x^2-9}{x-5}, x \right\} dx &= \int_{-1}^{9/5} \frac{x^2-9}{x-5} dx + \int_{9/5}^3 x dx \\ &= \int_{-1}^{9/5} \left( x+5 + \frac{16}{x-5} \right) dx + \frac{x^2}{2} \Big|_{9/5}^3 \\ &= \frac{28}{25} + 14 + 16 \ln \left( \frac{8}{15} \right) + \frac{72}{25} = 18 + 16 \ln \left( \frac{8}{15} \right) \end{aligned}$$

Clearly  $\alpha_1 = 18$  and  $\alpha_2 = 16$ , so  $\alpha_1 + \alpha_2 = 34$ .

## Question91

The integral  $\int \frac{\left(1 - \frac{1}{\sqrt{3}}\right)(\cos x - \sin x)}{\left(1 + \frac{2}{\sqrt{3}} \sin 2x\right)} dx$  is equal to

[26-Jul-2022-Shift-2]

Options:

A.  $\frac{1}{2} \log_e \left| \frac{\tan\left(\frac{x}{2} + \frac{\pi}{12}\right)}{\tan\left(\frac{x}{2} + \frac{\pi}{6}\right)} \right| + C$

B.  $\frac{1}{2} \log_e \left| \frac{\tan\left(\frac{x}{2} + \frac{\pi}{6}\right)}{\tan\left(\frac{x}{2} + \frac{\pi}{3}\right)} \right| + C$

C.  $\log_e \left| \frac{\tan\left(\frac{x}{2} + \frac{\pi}{6}\right)}{\tan\left(\frac{x}{2} + \frac{\pi}{12}\right)} \right| + C$

D.  $\frac{1}{2} \log_e \left| \frac{\tan\left(\frac{x}{2} - \frac{\pi}{12}\right)}{\tan\left(\frac{x}{2} - \frac{\pi}{6}\right)} \right| + C$

Answer: A

Solution:

$$\begin{aligned} &= \int \frac{\left(1 - \frac{1}{\sqrt{3}}\right)(\cos x - \sin x)}{\left(1 + \frac{2}{\sqrt{3}} \sin 2x\right)} dx \\ &= \int \frac{\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) \sqrt{2} \sin\left(\frac{\pi}{4} - x\right)}{\left(\frac{2}{\sqrt{3}}\right) \left(\sin \frac{\pi}{3} + \sin 2x\right)} dx \\ &= \int \frac{\frac{(\sqrt{3}-1)}{\sqrt{2}} \sin\left(\frac{\pi}{4} - x\right)}{\left(\sin \frac{\pi}{3} + \sin 2x\right)} dx \end{aligned}$$

$$\begin{aligned}
&= \int \frac{\frac{\sqrt{3}-1}{2\sqrt{2}} \sin\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{6}+x\right) \cos\left(\frac{\pi}{6}-x\right)} dx \\
&= \frac{1}{2} \int \frac{2 \sin \frac{\pi}{12} \sin\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{6}+x\right) \cos\left(\frac{\pi}{6}-x\right)} dx \\
&= \frac{1}{2} \int \frac{\cos\left(\frac{\pi}{6}-x\right) - \cos\left(\frac{\pi}{3}-x\right)}{\sin\left(\frac{\pi}{6}+x\right) \cos\left(\frac{\pi}{6}-x\right)} dx \\
&= \frac{1}{2} \left[ \int \operatorname{cosec}\left(\frac{\pi}{6}+x\right) dx - \int \sec\left(\frac{\pi}{6}-x\right) dx \right] \\
&= \frac{1}{2} \left[ \ln \left| \tan\left(\frac{\pi}{12} + \frac{x}{2}\right) \right| - \int \operatorname{cosec}\left(\frac{\pi}{3}-x\right) dx \right] \\
&= \frac{1}{2} \left[ \ln \left| \tan\left(\frac{\pi}{12} + \frac{x}{2}\right) \right| - \ln \left| \frac{\pi}{6} + \frac{x}{2} \right| \right] + C \\
&= \frac{1}{2} \ln \left| \frac{\tan\left(\frac{\pi}{2} + \frac{x}{2}\right)}{\tan\left(\frac{\pi}{6} + \frac{x}{2}\right)} \right| + C
\end{aligned}$$

## Question92

For  $I(x) = \int \frac{\sec^2 x - 2022}{\sin^{2022} x} dx$ , if  $I\left(\frac{\pi}{4}\right) - 2^{1011}$ , then  
**[29-Jul-2022-Shift-2]**

Options:

A.  $3^{1010} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$

B.  $3^{1010} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

C.  $3^{1011} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$

D.  $3^{1011} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

Answer: A

Solution:

Solution:

Given,

$$\begin{aligned}
I(x) &= \int \frac{\sec^2 x - 2022}{\sin^{2022} x} dx \\
&= \int \frac{\sec^2 x}{\sin^{2022} x} dx - \int \frac{2022}{\sin^{2022} x} dx \\
&= \int \frac{1}{\sin^{2022} x} \cdot \sec^2 x dx - \int \frac{2022}{\sin^{2022} x} dx \\
&= \frac{1}{\sin^{2022} x} \cdot \tan x - \int \left( \frac{-2022}{\sin^{2023} x} \cdot \cos x \cdot \tan x \right) dx - \int \frac{2022}{\sin^{2022} x} dx + C \\
&= \frac{\tan x}{\sin^{2022} x} + \int \left( \frac{2022}{\sin^{2023} x} \cdot \cos x \cdot \frac{\sin x}{\cos x} \right) dx - \int \frac{2022}{\sin^{2022} x} dx + C \\
&= \frac{\tan x}{\sin^{2022} x} + \int \frac{2022}{\sin^{2022} x} dx - \int \frac{2022}{\sin^{2022} x} dx \\
&= \frac{\tan x}{\sin^{2022} x} + C
\end{aligned}$$



$$\text{Given, } I\left(\frac{\pi}{4}\right) = 2^{1011} \therefore I\left(\frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{4}\right)}{\left(\sin\frac{\pi}{4}\right)^{2022}} + C$$

$$\Rightarrow 2^{1011} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^{2022}} + C$$

$$\Rightarrow C = 2^{1011} - 2^{1011} = 0$$

$$\therefore I(x) = \frac{\tan x}{\sin^{2022} x}$$

$$\therefore I\left(\frac{\pi}{3}\right) = \frac{\tan\frac{\pi}{3}}{\left(\sin\frac{\pi}{3}\right)^{2022}} = \frac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)^{2022}}$$

$$I\left(\frac{\pi}{6}\right) = \frac{\frac{1}{\sqrt{3}}}{\left(\frac{1}{2}\right)^{2022}} = \frac{1}{\sqrt{3}} \times (2)^{2022}$$

From option (A),

$$\begin{aligned} & 3^{1010} \cdot I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) \\ &= 3^{1010} \cdot \sqrt{3} \cdot \left(\frac{2}{\sqrt{3}}\right)^{2022} - \frac{(2)^{2022}}{\sqrt{3}} \\ &= 3^{1010} \cdot \sqrt{3} \times \frac{2^{2022}}{3^{1011}} - \frac{2^{2022}}{\sqrt{3}} \\ &= \frac{2^{2022}}{\sqrt{3}} - \frac{2^{2022}}{\sqrt{3}} = 0 \end{aligned}$$

## Question93

For any real number  $x$ , let  $[x]$  denote the largest integer less than equal to  $x$ . Let  $f$  be a real valued

function defined on the interval  $[-10, 10]$  by  $f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}$ .

Then the value of  $\frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx$  is :

[25-Jul-2022-Shift-1]

Options:

- A. 4
- B. 2
- C. 1
- D. 0

Answer: A

Solution:

Solution:

Case 1:

Let  $0 \leq x < 1$

then  $[x] = 0$ , which is even

$$\therefore f(x) = 1 + [x] - x$$

$$= 1 + 0 - x$$

$$= 1 - x$$

Case 2 :

Let  $1 \leq x < 2$

then  $[x] = 1$ , which is odd

$$\therefore f(x) = x - [x]$$

$$= x - 1$$

Case 3 :

Let  $2 \leq x < 3$  then  $[x] = 2$ , which is even

$$\begin{aligned} \therefore f(x) &= 1 + [x] - x \\ &= 1 + 2 - x \\ &= 3 - x \end{aligned}$$

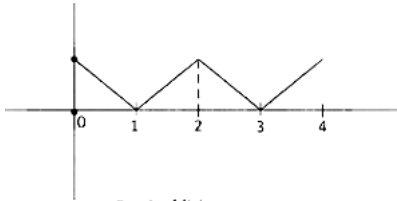
Case 4 :

Let  $3 \leq x < 4$

then  $[x] = 3$ , which is odd

$$\begin{aligned} \therefore f(x) &= x - [x] \\ &= x - 3 \end{aligned}$$

$$\therefore f(x) = \begin{cases} 1-x & 0 \leq x < 1 \\ x-1 & 1 \leq x < 2 \\ 3-x & 2 \leq x < 3 \\ x-3 & 3 \leq x < 4. \end{cases}$$



$\therefore f(x)$  is periodic and period of  $f(x) = 2$

And period of  $\cos \pi x = \frac{2\pi}{\pi} = 2$

$\therefore$  Period of  $f(x) \cos \pi x = 2$

Now,

$$\begin{aligned} I &= \frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx \\ &= \frac{\pi^2}{10} \int_{-10}^{-10+10 \times 2} f(x) \cos \pi x \, dx \\ &= \frac{\pi^2}{10} \int_0^{10 \times 2} f(x) \cos \pi x \, dx \\ &= \frac{\pi^2}{10} \times 10 \int_0^2 f(x) \cos \pi x \, dx \\ &= \pi^2 \int_0^2 f(x) \cos \pi x \, dx \\ \therefore I &= \pi^2 \left[ \int_0^1 f(x) \cos \pi x \, dx + \int_1^2 f(x) \cos \pi x \, dx \right] \\ &= \pi^2 \left[ \int_0^1 (1-x) \cos \pi x \, dx + \int_1^2 (x-1) \cos \pi x \, dx \right] \\ &= \pi^2 \left[ \int_0^1 \cos \pi x \, dx - \int_0^1 x \cos \pi x \, dx + \int_1^2 x \cos \pi x \, dx - \int_1^2 \cos \pi x \, dx \right] \\ &= \pi^2 \left[ \frac{1}{\pi} [\sin \pi x]_0^1 - \int_0^1 x \cos \pi x \, dx + \int_1^2 x \cos \pi x \, dx - \frac{1}{\pi} [\sin \pi x]_1^2 \right] \\ &= \pi^2 \left[ 0 - \int_0^1 x \cos \pi x \, dx + \int_1^2 x \cos \pi x \, dx - 0 \right] \\ &= \pi^2 \left[ - \left[ x \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x \right]_0^1 + \left[ x \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x \right]_1^2 \right] \left[ \text{As } \int x \cos \pi x \, dx = x \cdot \int \cos \pi x - \int \left( 1 \cdot \frac{\sin \pi x}{\pi} \right) dx = x \cdot \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x + c \right] \\ &= \pi^2 \left[ - \left[ \left( 1 \cdot \frac{\sin \pi}{\pi} + \frac{1}{\pi^2} \cdot \cos \pi \right) - \left( 0 + \frac{1}{\pi^2} \cdot \cos 0 \right) \right] + \left[ \left( 2 \cdot \frac{\sin 2\pi}{\pi} + \frac{1}{\pi^2} \cos 2\pi \right) - \left( 1 \cdot \frac{\sin \pi}{\pi} + \frac{1}{\pi^2} \cos \pi \right) \right] \right] \\ &= \pi^2 \left[ - \left\{ \left( -\frac{1}{\pi^2} \right) - \left( \frac{1}{\pi^2} \right) \right\} + \left\{ \left( \frac{1}{\pi^2} \right) - \left( -\frac{1}{\pi^2} \right) \right\} \right] \\ &= \pi^2 \left[ - \left( -\frac{2}{\pi^2} \right) + \frac{2}{\pi^2} \right] \\ &= \pi^2 \left[ \frac{2}{\pi^2} + \frac{2}{\pi^2} \right] \\ &= \pi^2 \times \frac{4}{\pi^2} \\ &= 4 \end{aligned}$$

## Question 94

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \left( \frac{1}{\sqrt{1 - \frac{1}{2^n}}} + \frac{1}{\sqrt{1 - \frac{2}{2^n}}} + \frac{1}{\sqrt{1 - \frac{3}{2^n}}} + \dots + \frac{1}{\sqrt{1 - \frac{2^n - 1}{2^n}}} \right)$$

is equal to  
[25-Jul-2022-Shift-2]

Options:

- A.  $\frac{1}{2}$
- B. 1
- C. 2
- D. -2

Answer: C

Solution:

Solution:

$$I = \lim_{n \rightarrow \infty} \frac{1}{2^n} \left( \frac{1}{\sqrt{1 - \frac{1}{2^n}}} + \frac{1}{\sqrt{1 - \frac{2}{2^n}}} + \frac{1}{\sqrt{1 - \frac{3}{2^n}}} + \dots + \frac{1}{\sqrt{1 - \frac{2^n - 1}{2^n}}} \right)$$

Let  $2^n = t$  and if  $n \rightarrow \infty$  then  $t \rightarrow \infty$

$$I = \lim_{n \rightarrow \infty} \frac{1}{t} \left( \sum_{r=1}^{t-1} \frac{1}{\sqrt{1 - \frac{r}{t}}} \right)$$

$$I = \int_0^1 \frac{dx}{\sqrt{1-x}} = \int_0^1 \frac{dx}{\sqrt{x}} \quad \left( \int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$
$$= \left[ 2x^{\frac{1}{2}} \right]_0^1 = 2$$

## Question95

Let  $|t|$  denote the greatest integer less than or equal to  $t$ . Then the value of the integral

$\int_{-3}^{101} ([\sin(\pi x)] + e^{[\cos(2\pi x)]}) dx$  is equal to

[25-Jul-2022-Shift-2]

Options:

- A.  $\frac{52(1-e)}{e}$
- B.  $\frac{52}{e}$
- C.  $\frac{52(2+e)}{e}$
- D.  $\frac{104}{e}$

Answer: B

Solution:

Solution:

$$I = \int_{-3}^{101} ([\sin(\pi x)] + e^{[\cos(2\pi x)]}) dx$$

$[\sin \pi x]$  is periodic with period 2 and  $e^{[\cos(2\pi x)]}$  is periodic with period 1.

So,

$$I = 52 \int_0^2 ([\sin \pi x] + e^{[\cos 2\pi x]}) dx$$

$$= 52 \left\{ \int_1^2 -1 dx + \int_{\frac{1}{4}}^{\frac{3}{4}} e^{-1} dx + \int_{\frac{5}{4}}^{\frac{7}{4}} e^{-1} dx + \int_0^{\frac{1}{4}} e^0 dx + \int_{\frac{3}{4}}^{\frac{5}{4}} e^0 dx + \int_{\frac{7}{4}}^2 e^0 dx \right\}$$

$$= \frac{52}{e}$$

## Question96

Let  $f$  be a twice differentiable function on  $C$ . If  $f'(0) = 4$  and  $f(x) + \int_0^x (x-t)f'(t)dt = (e^{2x} + e^{-2x}) \cos 2x + \frac{2}{a}x$ , then  $(2a+1)^5 a^2$  is equal to \_\_\_\_  
[25-Jul-2022-Shift-2]

Answer: 8

## Question97

Let  $a_n = \int_{-1}^n \left( 1 + \frac{x}{2} + \frac{x^2}{3} + \dots + \frac{x^{n-1}}{n} \right) dx$  for every  $n \in \mathbb{N}$ . Then the sum of all the elements of the set  $\{n \in \mathbb{N} : a_n \in (2, 30)\}$  is \_\_\_\_  
[25-Jul-2022-Shift-2]

Answer: 5

Solution:

Solution:

$$\begin{aligned} \because a_n &= \int_{-1}^n \left( 1 + \frac{x}{2} + \frac{x^2}{3} + \dots + \frac{x^{n-1}}{n} \right) dx \\ &= \left[ x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n} \right]_{-1}^n \\ a_n &= \frac{n+1}{1^2} + \frac{n^2-1}{2^2} + \frac{n^3+1}{3^2} + \frac{n^4-1}{4^2} + \dots + \frac{n^n + (-1)^{n+1}}{n^2} \end{aligned}$$

$$\text{Here, } a_1 = 2, a_2 = \frac{2+1}{1} + \frac{2^2-1}{2} = 3 + \frac{3}{2} = \frac{9}{2}$$

$$a_3 = 4 + 2 + \frac{28}{9} = \frac{100}{9}$$

$$a_4 = 5 + \frac{15}{4} + \frac{65}{9} + \frac{255}{16} > 31$$

$\therefore$  The required set is  $\{2, 3\}$ .  $\therefore a_n \in (2, 30)$

$\therefore$  Sum of elements = 5.

## Question98

If  $a = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2n}{n^2+k^2}$  and  $f(x) = \sqrt{\frac{1-\cos x}{1+\cos x}}$ ,  $x \in (0, 1)$ , then :

[26-Jul-2022-Shift-1]

Options:

A.  $2\sqrt{2}f\left(\frac{a}{2}\right) = f'\left(\frac{a}{2}\right)$

B.  $f\left(\frac{a}{2}\right)f'\left(\frac{a}{2}\right) = \sqrt{2}$

C.  $\sqrt{2}f\left(\frac{a}{2}\right) = f'\left(\frac{a}{2}\right)$

D.  $f\left(\frac{a}{2}\right) = \sqrt{2}f'\left(\frac{a}{2}\right)$

Answer: C

Solution:

$$a = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2n}{n^2+k^2}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{2}{1+\left(\frac{k}{n}\right)^2}$$

$$a = \int_0^1 \frac{2}{1+x^2} dx = 2 \tan^{-1} x \Big|_0^1 = \frac{\pi}{2}$$

$$f(x) = \sqrt{\frac{1-\cos x}{1+\cos x}}, x \in (0, 1)$$

$$f(x) = \frac{1-\cos x}{\sin x} = \operatorname{cosec} x - \cot x$$

$$f'(x) = \operatorname{cosec}^2 x - \operatorname{cosec} x \cot x$$

$$\left. \begin{aligned} f\left(\frac{a}{2}\right) &= f\left(\frac{\pi}{4}\right) = \sqrt{2}-1 \\ f'\left(\frac{a}{2}\right) &= f'\left(\frac{\pi}{4}\right) = 2-\sqrt{2} \end{aligned} \right\} f'\left(\frac{a}{2}\right) = \sqrt{2} \cdot f\left(\frac{a}{2}\right)$$

## Question99

If  $n(2n+1) \int_0^1 (1-x^n)^{2n} dx = 1177 \int_0^1 (1-x^n)^{2n+1} dx$ , then  $n \in \mathbb{N}$  is equal to \_\_\_\_\_.

[26-Jul-2022-Shift-1]

Answer: 24

Solution:

$$\int_0^1 (1-x^n)^{2n+1} dx = \int_0^1 1 \cdot (1-x^n)^{2n+1} dx$$

$$= \left[ (1-x^n)^{2n+1} \cdot x \right]_0^1 - \int_0^1 x \cdot (2n+1)(1-x^n)^{2n} \cdot -nx^{n-1} dx$$

$$= n(2n+1) \int_0^1 (1-(1-x^n))(1-x^n)^{2n} dx$$

$$= n(2n+1) \int_0^1 (1-x^n)^{2n} dx - n(2n+1) \int_0^1 (1-x^n)^{2n+1} dx$$

$$(1 + n(2n + 1)) \int_0^1 (1 - x^n)^{2n+1} dx = n(2n + 1) \int_0^1 (1 - x^n)^{2n} dx$$

$$(2n^2 + n + 1) \int_0^1 (1 - x^n)^{2n+1} dx = 1177 \int_0^1 (1 - x^n)^{2n} dx$$

$$\therefore 2n^2 + n + 1 = 1177$$

$$2n^2 + n - 1176 = 0$$

$$\therefore n = 24 \text{ or } -\frac{49}{2}$$

$$\therefore n = 24$$

---

## Question 100

$\int_0^{20\pi} (|\sin x| + |\cos x|)^2 dx$  is equal to  
[26-Jul-2022-Shift-2]

Options:

A.  $10(\pi + 4)$

B.  $10(\pi + 2)$

C.  $20(\pi - 2)$

D.  $20(\pi + 2)$

Answer: D

Solution:

$$I = \int_0^{20\pi} (|\sin x| + |\cos x|)^2 dx$$

$$= 20 \int_0^\pi (1 + |\sin 2x|) dx$$

$$= 40 \int_0^{\frac{\pi}{2}} (1 + \sin 2x) dx$$

$$= 40 \left( x - \frac{\cos 2x}{2} \right) \Big|_0^{\frac{\pi}{2}}$$

$$= 40 \left( \frac{\pi}{2} + \frac{1}{2} + \frac{1}{2} \right) = 20(\pi + 2)$$

---





# Question 101

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function defined as

$f(x) = a \sin\left(\frac{\pi[x]}{2}\right) + [2 - x]$ ,  $a \in \mathbb{R}$  where  $[t]$  is the greatest integer less than or equal to  $t$ . If  $\lim_{x \rightarrow -1} f(x)$  exists, then the value of  $\int_0^4 f(x) dx$  is equal to

[27-Jul-2022-Shift-1]

Options:

- A. -1
- B. -2
- C. 1
- D. 2

Answer: B

Solution:

Solution:

$$f(x) = a \sin\left(\frac{\pi[x]}{2}\right) + [2 - x], a \in \mathbb{R}$$

Now,

$$\because \lim_{x \rightarrow -1} f(x) \text{ exist}$$

$$\therefore \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x)$$

$$\Rightarrow a \sin\left(\frac{-2\pi}{2}\right) + 3 = a \sin\left(\frac{-\pi}{2}\right) + 2$$

$$\Rightarrow -a = 1 \Rightarrow a = -1$$

$$\begin{aligned} \text{Now, } \int_0^4 f(x) dx &= \int_0^4 \left(-\sin\left(\frac{\pi[x]}{2}\right) + [2 - x]\right) dx \\ &= \int_0^1 1 dx + \int_1^2 -1 dx + \int_2^3 -1 dx + \int_3^4 (1 - 2) dx \\ &= 1 - 1 - 1 - 1 = -2 \end{aligned}$$

-----

## Question102

Let  $f(x) = 2 + |x| - |x - 1| + |x + 1|, x \in \mathbb{R}$ .

Consider

$$(S1): f' \left( -\frac{3}{2} \right) + f' \left( -\frac{1}{2} \right) + f' \left( \frac{1}{2} \right) + f' \left( \frac{3}{2} \right) = 2$$

$$(S2) : \int_{-2}^2 f(x) dx = 12$$

Then,

[27-Jul-2022-Shift-2]

Options:

- A. both (S1) and (S2) are correct
- B. both (S1) and (S2) are wrong
- C. only (S1) is correct
- D. only (S2) is correct

Answer: D

Solution:

Solution:

$$f(x) = 2 + |x| - |x - 1| + |x + 1|, x \in \mathbb{R}$$

$$\therefore f(x) = \begin{cases} -x & , \quad x < -1 \\ x+2 & , \quad -1 \leq x < 0 \\ 3x+2 & , \quad 0 \leq x < 1 \\ x+4 & , \quad x \geq 1. \end{cases}$$

$$\therefore f' \left( -\frac{3}{2} \right) + f' \left( -\frac{1}{2} \right) + f' \left( \frac{1}{2} \right) + f' \left( \frac{3}{2} \right) = -1 + 1 + 3 + 1 = 4$$

$$\begin{aligned} \text{and } \int_{-2}^2 f(x) dx &= \int_{-2}^{-1} f(x) dx + \int_{-1}^0 f(x) dx + \int_{0}^1 f(x) dx + \int_1^2 f(x) dx \\ &= \left[ -\frac{x^2}{2} \right]_{-2}^{-1} + \left[ \frac{(x+2)^2}{2} \right]_{-1}^0 + \left[ \frac{(3x+2)^2}{6} \right]_0^1 + \left[ \frac{(x+4)^2}{2} \right]_1^2 \\ &= \frac{3}{2} + \frac{3}{2} + \frac{7}{2} + \frac{11}{2} = \frac{24}{2} = 12 \end{aligned}$$

$\therefore$  Only (S2) is correct

---

## Question103

$\int_0^2 \left( |2x^2 - 3x| + \left[ x - \frac{1}{2} \right] \right) dx$ , where  $[t]$  is the greatest integer function, is equal to:  
**[27-Jul-2022-Shift-2]**

**Options:**

A.  $\frac{7}{6}$

B.  $\frac{19}{12}$

C.  $\frac{31}{12}$

D.  $\frac{3}{2}$

**Answer: B**

**Solution:**

**Solution:**

$$\begin{aligned} & \int_0^2 |2x^2 - 3x| dx + \int_0^2 \left[ x - \frac{1}{2} \right] dx \\ &= \int_0^{3/2} (3x - 2x^2) dx + \int_{3/2}^2 (2x^2 - 3x) dx + \int_0^{1/2} -1 dx + \int_{1/2}^{3/2} 0 dx + \int_{3/2}^2 1 dx \\ &= \left( \frac{3x^2}{2} - \frac{2x^3}{3} \right) \Big|_0^{3/2} + \left( \frac{2x^3}{3} - \frac{3x^2}{2} \right) \Big|_{3/2}^2 - \frac{1}{2} + \frac{1}{2} \\ &= \left( \frac{27}{8} - \frac{27}{12} \right) + \left( \frac{16}{3} - 6 - \frac{27}{12} + \frac{27}{8} \right) \\ &= \frac{19}{12} \end{aligned}$$

## Question 104

Let  $f(x) = \min\{[x - 1], [x - 2], \dots, [x - 10]\}$  where  $[t]$  denotes the greatest integer  $\leq t$ . Then  $\int_0^{10} f(x) dx + \int_0^{10} (f(x))^2 dx + \int_0^{10} |f(x)| dx$  is equal to \_\_\_\_\_.

**[27-Jul-2022-Shift-2]**

**Answer: 385**



On integrating we get :  $\ln y = \frac{1}{2} \ln x + \ln c \therefore f(1) = \sqrt{3}$  then  $c = \sqrt{3}$

$\therefore (\alpha, 6)$  lies on

$$\therefore y = \sqrt{3x}$$

$$\therefore 6 = \sqrt{3\alpha} \Rightarrow \alpha = 12$$

---

## Question 106

If  $\int_0^{\sqrt{3}} \frac{15x^3}{\sqrt{1+x^2} + \sqrt{(1+x^2)^3}} dx = \alpha\sqrt{2} + \beta\sqrt{3}$ , where  $\alpha, \beta$  are integers, then  $\alpha + \beta$  is equal to

**[28-Jul-2022-Shift-1]**

**Answer: 10**

**Solution:**

**Solution:**

$$\text{Put } x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{15 \tan^3 \theta \cdot \sec^2 \theta d\theta}{\sqrt{1 + \tan^2 \theta} + \sqrt{\sec^6 \theta}}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{15 \tan^2 \theta \sec^2 \theta d\theta}{\sec \theta \sqrt{1 + \sec \theta}}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{15(\sec^2 \theta - 1) \sec \theta \tan \theta d\theta}{(\sqrt{1 + \sec \theta})}$$

$$\text{Now put } 1 + \sec \theta = t^2$$

$$\Rightarrow \sec \theta \tan \theta d\theta = 2tdt$$

$$\Rightarrow I = \int_{\sqrt{2}}^{\sqrt{3}} \frac{15((t^2 - 1)^2 - 1)2tdt}{t}$$

$$\Rightarrow I = 30 \int_{\sqrt{2}}^{\sqrt{3}} (t^4 - 2t^2 + 1 - 1)dt$$

$$\Rightarrow I = 30 \int_{\sqrt{2}}^{\sqrt{3}} (t^4 - 2t^2)dt$$

$$\Rightarrow I = 30 \int_{\sqrt{2}}^{\sqrt{3}} (t^4 - 2t^2)dt$$

$$\Rightarrow I = 30 \left( \frac{t^5}{5} - \frac{2t^3}{3} \right) \Big|_{\sqrt{2}}^{\sqrt{3}}$$

$$= 30 \left[ \left( \frac{9\sqrt{3}}{5} - 2\sqrt{3} \right) - \left( \frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} \right) \right]$$

$$= (54\sqrt{3} - 60\sqrt{3}) - (24\sqrt{2} - 40\sqrt{2})$$



$$= 16\sqrt{2} - 6\sqrt{3}$$

$$\therefore \alpha = 16 \text{ and } \beta = -6$$

$$\alpha + \beta = 10$$


---

## Question107

Let  $I_n(x) = \int_0^x \frac{1}{(t^2+5)^n} dt$ ,  $n = 1, 2, 3, \dots$  Then :  
**[28-Jul-2022-Shift-2]**

**Options:**

A.  $50I_6 - 9I_5 - xI_5'$

B.  $50I_6 - 11I_5 - xI_5'$

C.  $50I_6 - 9I_5 - I_5'$

D.  $50I_6 - 11I_5 - I_5'$

**Answer: A**

**Solution:**

**Solution:**

$$I_n(x) = \int_0^x \frac{1}{(t^2+5)^n} dt$$

$$= \int_0^x \frac{1}{(t^2+5)^n} \times \int_0^1 dt$$

$$= \frac{t}{(t^2+5)^n} \Big|_0^x - \int_0^x \frac{-2nt}{(t^2+5)^{n+1}} \times t dt$$

$$= \frac{x}{(x^2+5)^n} + \int_0^x 2n \left( \frac{t^2+5-5}{(t^2+5)^{n+1}} \right) dt$$

$$I_n(x) = \frac{x}{(x^2+5)^n} + 2nI_n(x) - 10nI_{n+1}(x)$$

$$10nI_{n+1}(x) - (2n-1)I_n(x) = xI_n'(x)$$

For  $n = 5$

$$50I_6(x) - 9I_5(x) = xI_5'(x)$$


---

## Question108

The value of the integral  $\int_0^{\frac{\pi}{2}} 60 \frac{\sin(6x)}{\sin x} dx$  is equal to \_\_\_\_\_.

[28-Jul-2022-Shift-2]

**Answer: 104**

**Solution:**

**Solution:**

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} 60 \cdot \frac{\sin 6x}{\sin x} dx \\ &= 60 \cdot 2 \int_0^{\frac{\pi}{2}} (3 - 4 \sin^2 x)(4 \cos^2 x - 3) \cos x dx \\ &= 120 \int_0^{\frac{\pi}{2}} (3 - 4 \sin^2 x)(1 - 4 \sin^2 x) \cos x dx \end{aligned}$$

Let  $\sin x = t \Rightarrow \cos x dx = dt$

$$\begin{aligned} &= 120 \int_0^1 (3 - 4t^2)(1 - 4t^2) dt \\ &= 120 \int_0^1 (3 - 16t^2 + 16t^4) dt \\ &= 120 \left[ 3t - \frac{16t^3}{3} + \frac{16t^5}{5} \right]_0^1 \\ &= 104 \end{aligned}$$

---

## Question 109

The integral  $\int_0^{\frac{\pi}{2}} \frac{1}{3 + 2 \sin x + \cos x} dx$  is equal to:

[29-Jul-2022-Shift-1]

**Options:**

A.  $\tan^{-1}$

B.  $\tan^{-1}(2) - \frac{\pi}{4}$

C.  $\frac{1}{2} \tan^{-1}(2) - \frac{\pi}{8}$

D.  $\frac{1}{2}$

**Answer: B**

**Solution:**

**Solution:**

$$I = \int_0^{\pi/2} \frac{1}{3 + 2 \sin x + \cos x} dx$$
$$= \int_0^{\pi/2} \frac{(1 + \tan^2 x/2) dx}{3(1 + \tan^2 x/2) + 2(2 \tan x/2) + (1 - \tan^2 x/2)}$$

Let  $\tan x/2 = t \Rightarrow \sec^2 x/2 dx = 2 dt$

$$I = \int_0^1 \frac{2 dt}{4 + 2t^2 + 4t}$$
$$= \int_0^1 \frac{dt}{t^2 + 2t + 2} = \int_0^1 \frac{dt}{(t+1)^2 + 1}$$
$$= .\tan^{-1}(t+1) \Big|_0^1 = \tan^{-1}2 - \frac{\pi}{4}$$

---

## Question110

If  $f(\alpha) = \int_1^\alpha \frac{\log_{10} t}{1+t} dt$ ,  $\alpha > 0$ , then  $f(e^3) + f(e^{-3})$  is equal to :

[29-Jul-2022-Shift-1]

**Options:**

A. 9

B.  $\frac{9}{2}$

C.  $\frac{9}{\log_e(10)}$

D.  $\frac{9}{2\log_a(10)}$

**Answer: D**

**Solution:**



$$f(\alpha) = \int_1^{\alpha} \frac{\log_{10} t}{1+t} dt \dots\dots (i)$$

$$f\left(\frac{1}{\alpha}\right) = \int_1^{\frac{1}{\alpha}} \frac{\log_{10} t}{1+t} dt$$

Substituting  $t \rightarrow \frac{1}{p}$

$$f\left(\frac{1}{\alpha}\right) = \int_1^{\alpha} \frac{\log_{10}\left(\frac{1}{p}\right)}{1+\frac{1}{p}} \left(\frac{-1}{p^2}\right) dp$$

$$= \int_1^{\alpha} \frac{\log_{10} p}{p(p+1)} dp = \int_1^{\alpha} \left( \frac{\log_{10} t}{t} - \frac{\log_{10} t}{t+1} \right) dt \dots\dots (ii)$$

By (i) + (ii)

$$f(\alpha) + f\left(\frac{1}{\alpha}\right) = \int_1^{\alpha} \frac{\log_{10} t}{t} dt = \int_1^{\alpha} \frac{\ln t}{t} \cdot \log_{10} e dt$$

$$= \frac{(\ln \alpha)^2}{2 \log_e 10}$$

$$\alpha = e^3 \Rightarrow f(e^3) + f(e^{-3}) = \frac{9}{2 \log_e 10}$$

## Question 111

If  $[t]$  denotes the greatest integer  $\leq t$ , then the value of

$$\int_0^1 [2x - |3x^2 - 5x + 2| + 1] dx \text{ is:}$$

[29-Jul-2022-Shift-2]

Options:

A.  $\frac{\sqrt{37} + \sqrt{13} - 4}{6}$

B.  $\frac{\sqrt{37} - \sqrt{13} - 4}{6}$

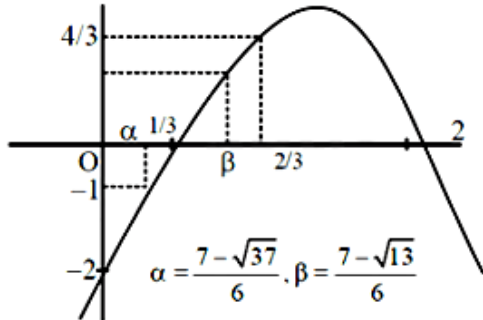
C.  $\frac{-\sqrt{37} - \sqrt{13} + 4}{6}$

D.  $\frac{-\sqrt{37} + \sqrt{13} + 4}{6}$

**Answer: A**

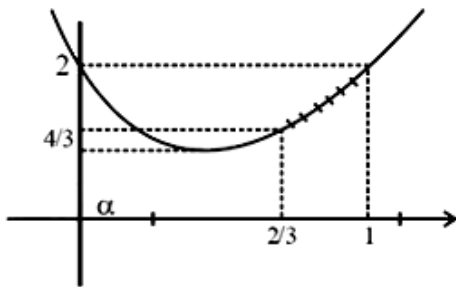
**Solution:**

$$\begin{aligned}
 I &= \int_0^1 [2x - 13x^2 - 3x - 2x + 2 | +1] dt \\
 I &= \int_0^1 [2x - |(3x-2)(x-1)|] dt + \int_0^1 1 dt \\
 I &= \int_0^{2/3} [(2x - (3x^2 - 5x + 2))] dt + \int_{2/3}^1 (2x + (3x^2 - 5x + 2)) dt + 1 \\
 I &= \int_0^{2/3} [-3x^2 + 7x - 2] dt + \int_{2/3}^1 (3x^2 - 3x + 2) dt + 1
 \end{aligned}$$



$$\begin{aligned}
 &\int_0^{\alpha} (-2) dt + \int_{\alpha}^{1/3} (-1) dt + \int_{1/3}^{\beta} 0 dt + \int_{\beta}^{2/3} 1 dt \\
 &= -2\alpha - \left(\frac{1}{3} - \alpha\right) + \frac{2}{3} - \beta = -\alpha - \beta + \frac{1}{3}
 \end{aligned}$$

$$y = 3x^2 - 3x + 2$$



When  $x \in \left(\frac{2}{3}, 1\right)$

$$3x^2 - 3x + 2 \in \left(\frac{4}{3}, 2\right)$$

$$[3x^2 - 3x + 2] = 1$$

$$\therefore \int_{2/3}^1 [3x^2 - 3x + 2] dt = 1 \left(1 - \frac{2}{3}\right) = \frac{1}{3}$$

$$\text{Hence } I = \left(\frac{1}{3} - (\alpha + \beta)\right) + \left(\frac{1}{3}\right) + 1$$

$$= \frac{5}{3} - \left(\frac{7 - \sqrt{37}}{6} + \frac{7 - \sqrt{13}}{6}\right)$$

$$= \frac{-2}{3} + \frac{\sqrt{37} + \sqrt{13}}{6}$$

$$= \frac{\sqrt{37} + \sqrt{13} - 4}{6}$$

## Question 112

The integral  $\int \frac{e^{3\log_e 2x} + 5e^{2\log_e 2x}}{e^{4\log_e x} + 5e^{3\log_e x} - 7e^{2\log_e x}} dx, x > 0$ , is equal to (where, c is a constant of integration)  
**[25 Feb 2021 Shift 2]**

**Options:**

- A.  $\log_e x^2 + 5x - 7 | +c$
- B.  $4\log_e x^2 + 5x - 7 | +c$
- C.  $\frac{1}{4}\log_e x^2 + 5x - 7 | +c$
- D.  $\log_e \sqrt{x^2 + 5x - 7} + c$

**Answer: B**

**Solution:**

**Solution:**

$$I = \int \frac{e^{3\log_e(2x)} + 5e^{2\log_e(2x)}}{e^{4\log_e(x)} + 5e^{3\log_e(x)} - 7e^{2\log_e(x)}} dx$$

$$= \int \frac{e^{\log_e(2x)^3} + 5e^{\log_e(2x)^2}}{e^{\log_e x^4} + 5e^{\log_e(x)^3} - 7e^{\log_e(x)^2}} dx$$

[using property  $a \log x = \log x^a$ ]

$$= \int \frac{8x^3 + 5(2x)^2}{x^4 + 5(x)^3 - 7x^2} dx \quad [\text{using property } a^{\log_a x} = x]$$

$$= \int \frac{8x^3 + 20x^2}{x^4 + 5x^3 - 7x^2} dx = \int \frac{4x^2(2x + 5)}{x^2(x^2 + 5x - 7)} dx$$

$$= \int \frac{4(2x + 5)}{x^2 + 5x - 7} dx$$

Let  $x^2 + 5x - 7 = t$ , then  $(2x + 5)dx = dt$

$$I = \int \frac{4dt}{t} = 4\log_e t + c$$

Put  $t = x^2 + 5x - 7$

$$I = 4\log_e |x^2 + 5x - 7| + c$$

## Question 113

**The value of the integral**

$$\int \frac{\sin \theta \sin 2\theta (\sin^6 \theta + \sin^4 \theta + \sin^2 \theta) \sqrt{2\sin^4 \theta + 3\sin^2 \theta + 6}}{1 - \cos 2\theta} d\theta$$

is (where,  $c$  is a constant of integration)  
 [25 Feb 2021 Shift 1]

Options:

A.  $\frac{1}{18}[11 - 18\sin^2\theta + 9\sin^4\theta - 2\sin^6\theta]^{\frac{3}{2}} + c$

B.  $\frac{1}{18}[9 - 2\cos^6\theta - 3\cos^4\theta - 6\cos^2\theta]^{\frac{3}{2}} + c$

C.  $\frac{1}{18}[9 - 2\sin^6\theta - 3\sin^4\theta - 6\sin^2\theta]^{\frac{3}{2}} + c$

D.  $\frac{1}{18}[11 - 18\cos^2\theta + 9\cos^4\theta - 2\cos^6\theta]^{\frac{3}{2}} + c$

Answer: D

Solution:

Solution:

Let

$$\int \left[ \frac{\sin \theta \cdot \sin 2\theta (\sin^6\theta + \sin^4\theta + \sin^2\theta) \sqrt{2\sin^4\theta + 3\sin^2\theta + 6}}{1 - \cos 2\theta} \right] d\theta$$

$$\because \sin 2A = 2 \sin A \cos A$$

$$\text{and } 1 - \cos 2A = 2\sin^2 A \quad \sin \theta \cdot 2 \sin \theta (\sin^6\theta + \sin^4\theta + \sin^2\theta)$$

$$I = \int \frac{\sqrt{2\sin^4\theta + 3\sin^2\theta + 6}}{2\sin^2\theta} d\theta$$

$$I = \int \frac{\cos \theta (\sin^6\theta + \sin^4\theta + \sin^2\theta)}{\sqrt{2\sin^4\theta + 3\sin^2\theta + 6}} d\theta$$

$$\sqrt{2\sin^4\theta + 3\sin^2\theta + 6} d\theta$$

$$= \int (t^6 + t^4 + t^2) \sqrt{2t^4 + 3t^2 + 6} dt$$

$$= \int (t^5 + t^3 + t) \sqrt{2t^6 + 3t^4 + 6t^2} dt$$

$$\text{Let } 2t^6 + 3t^4 + 6t^2 = z$$

$$\therefore dz = (12t^5 + 12t^3 + 12t) dt$$

$$\therefore dz = 12(t^5 + t^3 + t) dt$$

$$\text{Now, } \frac{1}{12} \int \sqrt{z} dz = \frac{1}{12} \times \frac{z^{3/2}}{3/2} + c$$

$$= \frac{1}{18} z^{3/2} + c$$

$$= \frac{1}{18} [2t^6 + 3t^4 + 6t^2]^{3/2} + c$$

$$= \frac{1}{18} [2\sin^6\theta + 3\sin^4\theta + 6\sin^2\theta]^{3/2} + c$$

$$= \frac{1}{18} [(1 - \cos^2\theta) \{2(1 - \cos^2\theta)^3 + 3 - 3\cos^2\theta + 6\}]^{3/2} + c$$

$$\begin{aligned}
&= \frac{1}{18}[(1 - \cos^2\theta)(2\cos^4\theta - 7\cos^2\theta + 11)]^{3/2} + c \\
&= \frac{1}{18}[-2\cos^6\theta + 9\cos^4\theta - 18\cos^2\theta + 11]^{3/2} + c \\
&= \frac{1}{18}[11 - 18\cos^2\theta + 9\cos^4\theta - 2\cos^6\theta]^{3/2}
\end{aligned}$$


---

## Question 114

For  $x > 0$ , if  $f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$ , then  $f(e) + f\left(\frac{1}{e}\right)$  is equal to

[26 Feb 2021 Shift 2]

Options:

- A. 1
- B. -1
- C.  $\frac{1}{2}$
- D. 0

Answer: C

Solution:

Solution:

$$f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$$

$$\text{Then, } f(e) = \int_1^{e^{\log_e e}} \frac{dt}{1+t} \dots (i)$$

$$\text{and } f\left(\frac{1}{e}\right) = \int_1^{\frac{1}{e^{\log_e \frac{1}{e}}}} \frac{\log_e t}{1+t} dt \dots (ii)$$

Let  $t = \frac{1}{u}$ ,  $dt = \frac{-1}{u^2} du$  and put in Eq. (ii), we get

$$f\left(\frac{1}{e}\right) = \int_1^{\frac{\log\left(\frac{1}{u}\right)}{1+\frac{1}{u}}} \cdot \frac{-1}{u^2} du = \int_1^{\frac{\log u}{u(u+1)}} du$$

Using change of variable

$$f\left(\frac{1}{e}\right) = \int_1^{\frac{\log t}{t(t+1)}} dt \dots (iii)$$

From Eqs. (i) and (iii), we get

$$f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\log t}{1+t} dt + \int_1^{\frac{\log t}{t(t+1)}} dt = \int_1^{e^{\log t}} \frac{gt}{t} dt$$

Take  $\log t = v$ , then  $\frac{1}{t} dt = dv$

$$f(e) + f\left(\frac{1}{e}\right) = \int_0^1 v \, dv = \left[\frac{v^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\therefore f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$$

## Question 115

If  $I_{m \cdot n} = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ , for  $m, n \geq 1$  and  $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \alpha I_{m \cdot n}$ ,  $\alpha \in \mathbf{R}$ ,

then  $\alpha$  equals \_\_\_\_\_.

[26 Feb 2021 Shift 2]

**Answer: 1**

**Solution:**

**Solution:**

$$\text{Given, } I_{mn} = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$\text{Using substitution put } x = \frac{1}{t+1}$$

$$\text{Then, } dx = \frac{-1}{(t+1)^2} dt$$

$$I_{mn} = \int_{\infty}^0 (-1) \frac{1}{(t+1)^{m-1}} \cdot \frac{t^{n-1}}{(t+1)^{n-1}} \cdot \frac{1}{(t+1)^2} dt$$

$$= - \int_{\infty}^0 \frac{t^{n-1}}{(t+1)^{m+n}} dt \dots (i)$$

Similarly,

$$I_{mn} = \int_0^1 x^{n-1}(1-x)^{m-1} dx \dots (ii)$$

$$\Rightarrow I_{mn} = \int_0^{\infty} \frac{t^{m-1}}{(t+1)^{m+n}} dt$$

From Eqs. (i) and (ii), we get

$$2I_{mn} = \int_0^{\infty} \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt$$

$$2I_{mn} = \int_0^{\infty} \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt + \int_1^{\infty} \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt \quad \text{Let } I_1 = \int_1^{\infty} \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt$$

$$\text{Let } t = \frac{1}{z}, \text{ then } dt = \frac{-1}{z^2} dz$$

$$I_1 = \int_1^0 (-1) \frac{\left(\frac{1}{z}\right)^{n-1} + \left(\frac{1}{z}\right)^{m-1}}{\left(\frac{1}{z} + 1\right)^{m+n}} \cdot \frac{1}{z^2} dz = - \int_1^0 \frac{z^{n-1} + z^{m-1}}{(z+1)^{m+n}} dz$$

$$\begin{aligned}
2I_{mn} &= \int_0^1 \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt \\
&- \int_1^0 \frac{z^{n-1} + z^{m-1}}{(z+1)^{m+n}} dz \\
&= 2 \int_0^1 \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt \\
\Rightarrow \alpha &= 1
\end{aligned}$$


---

## Question 116

The value of  $\sum_{n=1}^{100} \int_n^{n+1} e^{x-[x]} dx$ , where  $[x]$  is the greatest integer  $\leq x$ , is  
**[26 Feb 2021 Shift 1]**

**Options:**

- A.  $100(e-1)$
- B.  $100(1-e)$
- C.  $100e$
- D.  $100(1+e)$

**Answer: A**

**Solution:**

**Solution:**

Let 'x' be any real number, then  $x = [x] + \{x\}$ , where  $[x]$  is integer part of x and  $\{x\}$  is fractional part of x.  
Then,  $x - [x] = \{x\}$ , Also period of  $\{x\} = 1$

$$\text{Now, } \sum_{n=1}^{100} \int_n^{n+1} e^{x-[x]} dx = \sum_{n=1}^{100} \int_n^{n+1} e^{\{x\}} dx$$

[Difference between upper and lower limit is 1 unit]

$$\begin{aligned}
&= \int_0^1 e^{\{x\}} dx + \int_1^2 e^{\{x\}} dx + \dots + \int_{99}^{100} e^{\{x\}} dx \\
&= e^x \Big|_0^1 + e^{(x-1)} \Big|_1^2 + \dots + e^{(x-99)} \Big|_{99}^{100} \\
&= (e-1) + (e-1) + \dots + (e-1) = 100(e-1)
\end{aligned}$$


---

## Question 117

The value of  $\int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^x} dx$  is  
**[26 Feb 2021 Shift 1]**

**Options:**

A.  $\frac{\pi}{4}$

B.  $4\pi$

C.  $\frac{\pi}{2}$

D.  $2\pi$

**Answer: A**

**Solution:**

**Solution:**

$$\text{Let } I = \int_{-\pi/2}^{\pi} \frac{\cos^2 x}{1+3^x} dx \dots (i)$$

Using the property,  $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \frac{\cos^2(\pi/2 - \pi/2)}{1+3^{\pi/2 - \pi/2}} \\ &= \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^{-x}} dx \quad [:: \cos(-x) = \cos x] \end{aligned}$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{3^x \cos^2 x}{1+3^x} dx \dots (ii)$$

Adding Eqs. (i) and (ii),

$$\begin{aligned} 2I &= \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^x} dx + \int_{-\pi/2}^{\pi/2} \frac{3^x \cos^2 x}{1+3^x} dx \\ &= \int_{-\pi/2}^{\pi/2} \frac{(1+3^x) \cos^2 x}{1+3^x} dx = \int_{-\pi/2}^{\pi/2} \cos^2 x dx \\ &= \int_{-\pi/2}^{\pi/2} \frac{1 + \cos 2x}{2} dx \\ &= \frac{1}{2} \left[ x + \frac{\sin 2x}{2} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2} [\pi] \end{aligned}$$

$$\Rightarrow 2I = \pi/2 \Rightarrow I = \frac{\pi}{4}$$

---

## Question 118

The value of the integral  $\int_0^{\pi} |\sin 2x| dx$  is .....

[26 Feb 2021 Shift 1]





**Answer: 2**

**Solution:**

**Solution:**

$$\begin{aligned} \text{Let } I &= \int_0^{\pi} |\sin 2x| dx \\ &= 2 \int_0^{\pi/2} |\sin 2x| dx \quad [\because \sin 2x \text{ is periodic function}] \\ &= 2 \int_0^{\pi/2} \sin 2x dx \quad [\sin 2x \text{ is positive in range } (0, \pi/2)] \\ &= 2 \left[ \frac{-\cos 2x}{2} \right]_0^{\pi/2} \\ &= -[\cos \pi - \cos 0] = -(-1 - 1) = 2 \\ I &= 2 \end{aligned}$$

---

## Question 119

The value of  $\int_{-2}^2 |3x^2 - 3x - 6| dx$  is .....

[25 Feb 2021 Shift 2]

**Answer: 19**

**Solution:**

**Solution:**

$$\begin{aligned} \int_{-2}^2 |3x^2 - 3x - 6| dx &= I \text{ (say)} \\ I &= 3 \int_{-2}^2 |x^2 - x - 2| dx \\ &= 3 \left[ \int_{-2}^{-1} (x^2 - x - 2) dx + \int_{-1}^2 (-x^2 + x + 2) dx \right] \\ &= 3 \left[ \left( \frac{x^3}{3} - \frac{x^2}{2} - 2x \right)^{-1} - \left( \frac{x^3}{3} - \frac{x^2}{2} - 2x \right)^2 \right] \\ &= 19 \end{aligned}$$

---

## Question120

If  $I_n = \int_{\pi/4}^{\pi/2} \cot^n x dx$ , then

[25 Feb 2021 Shift 2]

Options:

A.  $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$  are in AP

B.  $I_2 + I_4, I_3 + I_5, I_4 + I_6$  are in AP

C.  $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$  are in GP

D.  $I_2 + I_4, (I_3 + I_5)^2, I_4 + I_6$  are in GP

Answer: A

Solution:

Solution:

$$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x (\cot^2 x) dx$$

$$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x \operatorname{cosec}^2 x dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x dx$$

$$I_n + I_{n-2} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x \cdot \operatorname{cosec}^2 x dx$$

Now, let  $\cot x = t$ , then  $\operatorname{cosec}^2 x dx = -dt$ , limit will be

$$I_n + I_{n-2} = \int_1^0 -t^{n-2} dt$$

$$= \left[ \frac{-(t)^{n-1}}{n-1} \right]_1^0 = - \left\{ \frac{0}{n-1} - \frac{(1)^{n-1}}{n-1} \right\}$$

$$I_n + I_{n-2} = \frac{1}{n-1}$$

Now, put  $n = 4$

$$\Rightarrow I_2 + I_4 = \frac{1}{3}, \text{ then } \frac{1}{I_2 + I_4} = 3 \dots (i)$$

Put  $n = 5$

$$\Rightarrow I_3 + I_5 = \frac{1}{4}, \text{ then } \frac{1}{I_3 + I_5} = 4 \dots (ii)$$

Put  $n = 6$

$$\Rightarrow I_4 + I_6 = \frac{1}{5}, \text{ then } \frac{1}{I_4 + I_6} = 5 \dots (iii)$$



Here, from Eqs. (i), (ii) and (iii), we conclude

$\frac{1}{I_2+I_4}$ ,  $\frac{1}{I_3+I_5}$  and  $\frac{1}{I_4+I_6}$  are in AP with common difference 1.

---

## Question 121

The value of  $\int_{-1}^1 x^2 e^{[x^3]} dx$ , where  $[t]$  denotes the greatest integer  $\leq t$ , is  
[25 Feb 2021 Shift 1]

Options:

A.  $\frac{e-1}{3e}$

B.  $\frac{e+1}{3}$

C.  $\frac{e+1}{3e}$

D.  $\frac{1}{3e}$

Answer: C

Solution:

Solution:

Given,  $\int_{-1}^1 x^2 e^{[x^3]} dx$ , where  $[t]$  is greatest integer function.

$$\because [x^3] = 0 \quad \forall x \in (0, 1)$$

$$\text{and } [x^3] = -1 \quad \forall x \in (-1, 0)$$

$$\text{So, } \int_{-1}^1 x^2 e^{[x^3]} dx = \int_{-1}^0 x^2 e^{-1} dx + \int_0^1 x^2 e^0 dx$$

$$= \frac{1}{e} \int_{-1}^0 x^2 dx + \int_0^1 x^2 dx = \frac{1}{e} \times \left[ \frac{x^3}{3} \right]_{-1}^0 + \left[ \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{e} \times \left[ 0 + \frac{1}{3} \right] + \left[ \frac{1}{3} \right] = \frac{1}{3e} + \frac{1}{3} = \left( \frac{1+e}{3e} \right)$$

---

## Question 122

Let  $f(x)$  be a differentiable function defined on  $[0, 2]$ , such that

$f'(x) = f'(2-x)$ , for all  $x \in (0, 2)$ ,  $f(0) = 1$  and  $f(2) = e^2$ . Then, the value of



$$\int_0^2 f(x) dx \text{ is}$$

[24 Feb 2021 Shift 2]

Options:

A.  $1 - e^2$

B.  $1 + e^2$

C.  $2(1 - e^2)$

D.  $2(1 + e^2)$

Answer: B

Solution:

Solution:

Given,  $f(0) = 1 \dots$  (i)

$f(2) = e^2 \dots$  (ii)

$f'(x) = f'(2-x)$

Integrating w.r.t.  $x$ ,

$f(x) = -f(2-x) + C$

Put  $x = 0$

$f(0) = -f(2) + C$

$\Rightarrow 1 = -e^2 + C$  [from Eqs. (i) and (ii)]

$\Rightarrow C = 1 + e^2$

$\therefore f(x) = -f(2-x) + 1 + e^2$

or  $f(x) + f(2-x) = 1 + e^2 \dots$  (iii)

Let  $I = \int_0^2 f(x) dx \dots$  (iv)

Also,  $I = \int_0^2 f(2-x) dx \dots$  (v)

Now, adding Eqs. (iv) and (v),

$2I = \int_0^2 [f(x) + f(2-x)] dx$

$2I = \int_0^2 (1 + e^2) dx$  [from Eq. (iii)]

$2I = 2(1 + e^2)$

$\therefore I = (1 + e^2)$

---

## Question 123

The value of the integral  $\int_1^3 [x^2 - 2x - 2] dx$ , where  $[x]$  denotes the greatest integer less than or equal to  $x$ , is

## [24 Feb 2021 Shift 2]

Options:

A.  $-\sqrt{2} - \sqrt{3} + 1$

B.  $-\sqrt{2} - \sqrt{3} - 1$

C.  $-5$

D.  $-4$

Answer: B

Solution:

Solution:

$$\text{Let } I = \int_1^3 [x^2 - 2x - 2] dx$$

$$= \int_1^3 [x^2 - 2x + 1 - 3] dx = \int_1^3 [(x-1)^2 - 3] dx$$

$$= \int_1^3 [(x-1)^2] dx + \int_1^3 -3 dx$$

Put  $x-1 = t$ ;  $dx = dt$ , when  $x = 1$ ,  $t = 0$  and  $x = 3$ ,  $t = 2$

$$\therefore I = -3[x]_1^3 + \int_0^2 [t^2] dt$$

$$= -6 + \int_0^1 0 dt + \int_1^{\sqrt{2}} 1 dt + \int_{\sqrt{2}}^{\sqrt{3}} 2 dt + \int_{\sqrt{3}}^2 3 dt$$

$$= -6 + (0) + (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})$$

$$= -6 + \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}$$

$$I = -1 - \sqrt{2} - \sqrt{3}$$

---

## Question 124

$$\lim_{x \rightarrow 0} \frac{\int_0^{x^2} (\sin \sqrt{t}) dt}{x^3} \text{ is equal to :}$$

24 Feb 2021 Shift 1

Options:

A.  $\frac{2}{3}$

B.  $\frac{3}{2}$



C. 0

D.  $\frac{1}{15}$

**Answer: A**

**Solution:**

**Solution:**

$$\lim_{x \rightarrow 0^+} \frac{\int_0^{x^2} \sin \sqrt{t} \, dt}{x^3} = \lim_{x \rightarrow 0^+} \frac{(\sin x)2x}{3x^2}$$
$$= \lim_{x \rightarrow 0^+} \left( \frac{\sin x}{x} \right) \times \frac{2}{3} = \frac{2}{3}$$

---

## Question 125

If  $\int_{-a}^a (|x| + |x - 2|) \, dx = 22$ , ( $a > 2$ ) and  $[x]$  denotes the greatest integer  $\leq x$ , then

$\int_a^{-a} (x + [x]) \, dx$  is equal to \_\_\_

24 Feb 2021 Shift 1

**Answer: 3**

**Solution:**

**Solution:**

$$\int_{-a}^0 (-2x + 2) \, dx + \int_0^2 (x + 2 - x) \, dx + \int_2^a (2x - 2) \, dx = 22$$

$$\Rightarrow x^2 - 2x \Big|_0^{-a} + 2x \Big|_0^2 + x^2 - 2x \Big|_2^a = 22$$

$$\Rightarrow a^2 + 2a + 4 + a^2 - 2a - (4 - 4) = 22$$

$$\Rightarrow 2a^2 = 18 \Rightarrow a = 3$$

$$\therefore \int_3^{-3} (x + [x]) \, dx = -(-3 - 2 - 1 + 1 + 2) = 3$$

---

## Question 126

The integral  $\int \frac{(2x-1) \cos \sqrt{(2x-1)^2+5}}{\sqrt{4x^2-4x+6}} dx$  is equal to (where, c is a constant of integration)  
**[18 Mar 2021 Shift 1]**

**Options:**

A.  $\frac{1}{2} \sin \sqrt{(2x-1)^2+5} + c$

B.  $\frac{1}{2} \cos \sqrt{(2x+1)^2+5} + c$

C.  $\frac{1}{2} \cos \sqrt{(2x-1)^2+5} + c$

D.  $\frac{1}{2} \sin \sqrt{(2x+1)^2+5} + c$

**Answer: A**

**Solution:**

**Solution:**

$$\begin{aligned} \text{Let } I &= \int \frac{(2x-1) \cos \sqrt{(2x-1)^2+5}}{\sqrt{4x^2-4x+6}} dx \\ &= \int \frac{(2x-1) \cos \sqrt{(2x-1)^2+5}}{\sqrt{(2x-1)^2+5}} dx \end{aligned}$$

$$\begin{aligned} \text{Putting } (2x-1)^2+5 &= z^2 \\ \Rightarrow 2(2x-1) \times 2 \cdot dx &= 2z dz \\ \Rightarrow (2x-1) dx &= \frac{1}{2} z dz \end{aligned}$$

$$\begin{aligned} \therefore I &= \int \frac{\cos z}{z} \cdot \frac{1}{2} z \cdot dz = \frac{1}{2} \int \cos z dz = \frac{1}{2} \sin z + C \\ &= \frac{1}{2} \sin \sqrt{(2x-1)^2+5} + c \end{aligned}$$

[Note You can also substitute  $\sqrt{(2x-1)^2+5} = z$  and then proceed.]

## Question 127

If  $f(x) = \int \frac{5x^8+7x^6}{(x^2+1+2x^7)^2} dx$ , ( $x \geq 0$ ),  $f(0) = 0$  and  $f(1) = \frac{1}{K}$ , then the value of K is  
**[18 Mar 2021 Shift 1]**

Answer: 4

Solution:

Solution:

$$\text{Let } I = \int \frac{5x^8 + 7x^6}{(x^2 + 1 + 2x^7)^2} dx$$

$$= \int \frac{5x^8 + 7x^6}{x^{14}(x^{-5} + x^{-7} + 2)^2} dx$$

$$= \int \frac{\frac{5x^8 + 7x^6}{x^{14}}}{(x^{-5} + x^{-7} + 2)^2} dx$$

$$\Rightarrow I = \int \frac{5x^{-6} + 7x^{-8}}{(x^{-5} + x^{-7} + 2)^2} dx$$

Putting  $x^{-5} + x^{-7} + 2 = z$

$$\Rightarrow -(5x^{-6} + 7x^{-8}) dx = dz$$

$$\therefore I = -\int \frac{dz}{z^2} = -\left(\frac{1}{-z}\right) + c$$

$$\Rightarrow \Rightarrow I = \frac{1}{x^{-5} + x^{-7} + 2} + c$$

$$\Rightarrow f(x) = \frac{x^7}{x^2 + 1 + 2x^7} + c$$

Given,  $f(0) = 0$

$$\Rightarrow c = 0$$

$$\therefore f(x) = \frac{x^7}{x^2 + 1 + 2x^7}$$

$$\therefore f(1) = \frac{1}{1 + 1 + 2}$$

$$= \frac{1}{4} = \frac{1}{K}$$

Hence,  $K = 4$ .

---

## Question 128

For real numbers  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ , if  $\int \frac{(x^2 - 1) + \tan^{-1}\left(\frac{x^2 + 1}{x}\right)}{(x^4 + 3x^2 + 1)\tan^{-1}\left(\frac{x^2 + 1}{x}\right)} dx$

$$= \alpha \log_e \left[ \tan^{-1}\left(\frac{x^2 + 1}{x}\right) \right] + \beta \tan^{-1}\left(\frac{\gamma(x^2 - 1)}{x}\right) + \delta \tan^{-1}\left(\frac{x^2 + 1}{x}\right) + C$$

where  $C$  is an arbitrary constant, then the value of  $10(\alpha + \beta\gamma + \delta)$  is equal to.....

[16 Mar 2021 Shift 2]



**Answer: 6**

**Solution:**

**Solution:**

$$\text{Let } I = \int \frac{(x^2 - 1) + \tan^{-1}\left(\frac{x^2 + 1}{x}\right)}{(x^4 + 3x^2 + 1)\tan^{-1}\left(\frac{x^2 + 1}{x}\right)} dx$$

$$\Rightarrow I = \int \frac{x^2 - 1}{(x^4 + 3x^2 + 1)\tan^{-1}\left(\frac{x^2 + 1}{x}\right)} dx + \int \frac{1}{x^4 + 3x^2 + 1} dx$$

$$\text{Again let } I_1 = \int \frac{x^2 - 1}{(x^4 + 3x^2 + 1)\tan^{-1}\left(\frac{x^2 + 1}{x}\right)} dx$$

$$\text{and } I_2 = \int \frac{dx}{x^4 + 3x^2 + 1}$$

$$\therefore I = I_1 + I_2 \dots (i)$$

$$\text{Now, } I_1 = \int \frac{(x^2 - 1)}{(x^4 + 3x^2 + 1)\tan^{-1}\left(\frac{x^2 + 1}{x}\right)} dx$$

$$\text{Let } \tan^{-1}\left(\frac{x^2 + 1}{x}\right) = t$$

$$\Rightarrow \frac{x^2 - 1}{(x^4 + 3x^2 + 1)} dx = dt$$

$$\therefore I_1 = \int \frac{dt}{t} = \log|t| + C_1 = \log\left|\tan^{-1}\left(\frac{x^2 + 1}{x}\right)\right| + C_1$$

$$I_2 = \int \frac{1}{x^4 + 3x^2 + 1} dx = \frac{1}{2} \int \frac{(x^2 + 1) - (x^2 - 1)}{x^4 + 3x^2 + 1} dx$$

$$= \frac{1}{2} \int \frac{x^2 + 1}{x^4 + 3x^2 + 1} dx - \frac{1}{2} \int \frac{x^2 - 1}{x^4 + 3x^2 + 1} dx$$

$$= \frac{1}{2} \int \frac{1 + 1/x^2}{x^2 + 3 + \frac{1}{x^2}} dx - \frac{1}{2} \int \frac{1 - 1/x^2}{x^2 + 3 + \frac{1}{x^2}} dx$$

$$I_2 = \frac{1}{2} \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 5} dx - \frac{1}{2} \int \frac{1 - 1/x^2}{\left(x + \frac{1}{x}\right)^2 + 1} dx$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{5}x}\right) - \frac{1}{2} \tan^{-1}\left(\frac{x^2 + 1}{x}\right) + C_2$$

$$I = \log\left|\tan^{-1}\left(\frac{x^2 + 1}{x}\right)\right| + \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{5}x}\right) - \frac{1}{2} \tan^{-1}\left(\frac{x^2 + 1}{x}\right) + C$$

$$= \alpha \log_e\left(\tan^{-1}\left(\frac{x^2 + 1}{x}\right)\right) + \beta \tan^{-1}\left(\frac{\gamma(x^2 - 1)}{x}\right) + \delta \tan^{-1}\left(\frac{x^2 + 1}{x}\right) + C \text{ (given)}$$



$$\therefore \alpha = 1, \beta = \frac{1}{2\sqrt{5}}, \gamma = \frac{1}{\sqrt{5}} \text{ and } \delta = -\frac{1}{2}$$

$\therefore$  Required value of  $10(\alpha + \beta\gamma + \delta)$

$$= 10\left(1 + \frac{1}{10} - \frac{1}{2}\right)$$

$$= 10\left(\frac{10+1-5}{10}\right)$$

$$= 6$$

## Question 129

Let  $f(x)$  and  $g(x)$  be two functions satisfying  $f(x^2) + g(4-x) = 4x^3$  and  $g(4-x) + g(x) = 0$  then the value of  $\int_{-4}^4 \int_{-1}^2 (x)^2 dx$  is .....

[18 Mar 2021 Shift 1]

**Answer: 512**

**Solution:**

**Solution:**

$$\text{Given, } f(x^2) + g(4-x) = 4x^3$$

$$\text{and } g(4-x) + g(x) = 0$$

$$\text{Let } I = \int_{-4}^4 f(x^2) dx$$

$$= 2 \int_0^4 f(x^2) dx$$

$$\Rightarrow I = 2 \cdot \int_0^4 [4x^3 - g(4-x)] dx$$

$$= 8 \int_0^4 x^3 dx - 2 \int_0^4 g(4-x) dx$$

$$= 8 \left(\frac{x^4}{4}\right)_0^4 - 2I_1 = 2(4^4 - 0^4) - 2I_1$$

$$= 2^9 - 2I_1$$

$$\text{where, } I_1 = \int_0^4 g(4-x) dx$$

$$\text{Now, } I_1 = \int_0^4 g(4-x) dx \dots (i)$$

$$\Rightarrow I_1 = \int_0^4 g \left[ 4 - (0+4-x) \right] dx \left[ \because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$\Rightarrow I_1 = \int_0^4 g(x) dx \dots (ii)$$

Adding Eqs. (i) and (ii),

$$2I_1 = \int_0^4 [g(x) + g(4-x)] dx$$

$$\Rightarrow 2I_1 = 0$$

$$\Rightarrow I_1 = 0 \quad (\because g(x) + g(4-x) = 0, \text{ given})$$

$$\therefore I = 2^9 - 2I_1$$

$$\Rightarrow I = 2^9 = 512$$


---

## Question 130

Let  $P(x)$  be a real polynomial of degree 3 which vanishes at  $x = -3$ . Let  $P(x)$  have local minima at  $x = 1$ , local maxima at  $x = -1$  and  $\int_{-1}^1 P(x) dx = 18$ , then the sum of all the coefficients of the polynomial  $P(x)$  is equal to  
[18 Mar 2021 Shift 2]

**Answer: 8**

**Solution:**

**Solution:**

$$\text{Let } P'(x) = a(x-1)(x+1)$$

$$\Rightarrow P'(x) = a(x^2 - 1)$$

$$\therefore P(x) = a \int (x^2 - 1) dx \Rightarrow P(x) = a \left( \frac{x^3}{3} - x \right) + C$$

According to the question,  $P(-3) = 0$

$$a \left( -\frac{27}{3} + 3 \right) + C = 0$$

$$\Rightarrow -6a + C = 0 \dots (i)$$

$$\text{Now, } \int_{-1}^1 \left( a \left( \frac{x^3}{3} - x \right) + C \right) dx = 18 \quad (\text{given})$$

$$\Rightarrow 2C = 18$$

$$\Rightarrow C = 9 \dots (ii)$$

From Eqs. (i) and (ii),

$$-6a + 9 = 0 \Rightarrow a = \frac{3}{2}$$

$$\therefore P(x) = \frac{3}{2} \left( \frac{x^3}{3} - x \right) + 9$$

$$\therefore \text{Sum of the all coefficient} = \frac{1}{2} - \frac{3}{2} + 9 = 8$$


---

## Question 131

Which of the following statements is correct for the function  $g(\alpha)$  for  $\alpha \in \mathbb{R}$ ,

$$\text{such that } g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx$$

[17 Mar 2021 Shift 1]

Options:

- A.  $g(\alpha)$  is a strictly increasing function
- B.  $g(\alpha)$  has an inflection point at  $\alpha = -\frac{1}{2}$
- C.  $g(\alpha)$  is a strictly decreasing function
- D.  $g(\alpha)$  is an even function

Answer: D

Solution:

Solution:

$$g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx \dots (i)$$

$$\text{Applying } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^\alpha(\pi/2-x)}{\cos^\alpha\left(\frac{\pi}{2}-x\right) + \sin^\alpha\left(\frac{\pi}{2}-x\right)} dx$$

$$g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\cos^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx \dots (ii)$$

Adding Eqs. (i) and (ii),

$$2g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^\alpha x + \cos^\alpha x}{\sin^\alpha x + \cos^\alpha x} dx$$

$$2g(\alpha) = \int_{\pi/6}^{\pi/3} 1 \cdot dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore g(\alpha) = \frac{\pi}{12}$$

$g(\alpha)$  is constant function.

$\therefore$  It is even function.

## Question 132

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = e^{-x} \sin x$ . If  $F : [0, 1] \rightarrow \mathbb{R}$  is a differentiable function, such that  $F(x) = \int_0^x f(t) dt$ , then the value of  $\int_0^1 [F'(x) + f(x)] e^x dx$  lies in the interval

[17 Mar 2021 Shift 2]

### Options:

A.  $\left[ \frac{327}{360}, \frac{329}{360} \right]$

B.  $\left[ \frac{330}{360}, \frac{331}{360} \right]$

C.  $\frac{331}{360}, \frac{334}{360}$

D.  $\left[ \frac{335}{360}, \frac{336}{360} \right]$

**Answer: B**

### Solution:

#### Solution:

Given,  $f(x) = e^{-x} \cdot \sin x$

and  $F(x) = \int_0^x f(t) dt$

$\therefore F(x)$  is differentiable function.

$\therefore F'(x) = f(x) \times 1 - f(0) \times 0$  (using Newton-Leibnitz rule)

$\Rightarrow F'(x) = f(x) \dots (i)$

Let  $I = \int_0^1 [F'(x) + f(x)] e^x dx$

$= \int_0^1 [f(x) + f(x)] e^x dx = \int_0^1 2 \cdot f(x) \cdot e^x dx$  [ from Eq. (i) ]

$\Rightarrow I = 2 \cdot \int_0^1 f(x) \cdot e^x dx$

$= 2 \cdot \int_0^1 e^{-x} \sin x \cdot e^x dx$

$= 2 \int_0^1 \sin x dx = 2[-\cos x]_0^1$

$= 2[-(\cos 1 - \cos 0)] = 2(1 - \cos 1)$

$\Rightarrow 1 = 2 \cdot \left[ 1 - \left( 1 - \frac{(1)^2}{2!} + \frac{(1)^4}{4!} - \frac{(1)^6}{6!} + \frac{(1)^8}{8!} - \dots \right) \right]$

[using expansion of  $\cos x$  i.e.,

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$  ]

$I = 2 \left[ 1 - 1 + \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \frac{1}{8!} + \dots \right]$

$\Rightarrow I = 2 \left( \frac{1}{2} - \frac{1}{24} + \frac{1}{720} - \dots \right)$

Now,  $I < 2 \left( \frac{1}{2} - \frac{1}{24} + \frac{1}{720} \right)$

$\Rightarrow I < \left( 1 - \frac{1}{12} + \frac{1}{360} \right) \Rightarrow I < \frac{360 - 30 + 1}{360}$

$\Rightarrow I < \frac{331}{360} \dots (ii)$

$$\text{Also, } I > 2 \left( \frac{1}{2} - \frac{1}{24} \right) \Rightarrow I > \left( 1 - \frac{1}{12} \right)$$

$$\Rightarrow I > \frac{11}{12} \Rightarrow I > \frac{11 \times 30}{12 \times 30}$$

$$\Rightarrow I > \frac{330}{360} \dots \text{(iii)}$$

From Eqs. (ii) and (iii), we get

$$\frac{330}{360} < 1 < \frac{331}{360}$$

## Question 133

If the integral  $\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$ , where  $\alpha, \beta, \gamma$  are integers and  $[x]$  denotes the greatest integer less than or equal to  $x$ , then the value of  $\alpha + \beta + \gamma$  is equal to

[17 Mar 2021 Shift 2]

Options:

- A. 0
- B. 20
- C. 25
- D. 10

Answer: A

Solution:

Solution:

$$\text{Let } I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx$$

$$\Rightarrow I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{\{x\}}} dx \quad [ \because x - [x] = \{x\} ]$$

From above integrand, we observe that  $\frac{[\sin 2\pi x]}{e^{\{x\}}}$  is a periodic function with period '1'.

$$\therefore I = 10 \int_0^1 \frac{[\sin 2\pi x]}{e^{\{x\}}} dx$$

[by the property of definite integral,

$$\int_0^{nT} f(x) dx = n \int_0^T f(x) dx$$

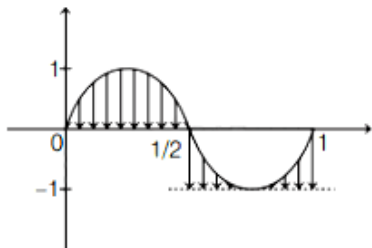
where  $f(x)$  is a periodic function with period  $= T$  ]

$$\Rightarrow I = 10 \cdot \int_0^1 \frac{[\sin 2\pi x]}{e^x} dx \quad [\because \{x\} = x, 0 \leq x < 1]$$

$$\Rightarrow I = 10 \left( \int_0^{1/2} \frac{[\sin 2\pi x]}{e^x} dx + \int_{1/2}^1 \frac{[\sin 2\pi x]}{e^x} dx \right)$$

$$\Rightarrow I = 10 \left( \int_0^{1/2} \frac{0}{e^x} dx + \int_{1/2}^1 \frac{(-1)}{e^x} dx \right)$$

$$\Rightarrow I = 10 \left( 0 - \int_{1/2}^1 e^{-x} dx \right) \Rightarrow I = -10 \left[ \frac{e^{-x}}{-1} \right]_{1/2}^1$$



$$\Rightarrow I = 10[e^{-1} - e^{-1/2}] \Rightarrow I = 10e^{-1} - 10e^{-1/2}$$

$$\Rightarrow I = 10e^{-1} + (-10) \cdot e^{-1/2} + 0 \dots (i)$$

Comparing Eq. (i) by  $\alpha e^{-1} + \beta e^{-1/2} + \gamma$ , we get

$$\alpha = 10, \beta = -10 \text{ and } \gamma = 0$$

$$\text{Hence, } \alpha + \beta + \gamma = 10 - 10 + 0$$

$$\Rightarrow \alpha + \beta + \gamma = 0$$

## Question 134

Let  $I_n = \int_1^e x^{19} (\log |x|)^n dx$ , where  $n \in \mathbb{N}$ . If  $(20) I_{10} = \alpha I_9 + \beta I_8$ , for natural numbers  $\alpha$  and  $\beta$ , then  $\alpha - \beta$  is equal to .....  
[17 Mar 2021 Shift 2]

**Answer: 1**

**Solution:**

**Solution:**

$$\text{Given, } I_n = \int_1^e x^{19} (\log |x|)^n dx$$

$$\Rightarrow I_n = \left[ \frac{x^{20}}{20} (\ln |x|)^n \right]_1^e - \int_1^e n \cdot \frac{(\ln |x|)^{n-1}}{x} \cdot \frac{x^{20}}{20} dx$$

(using integration by parts)

$$\Rightarrow I_n = \frac{e^{20}}{20} - \frac{n}{20} \int_1^e (\ln |x|)^{n-1} \cdot x^{19} dx$$

$$\Rightarrow I_n = \frac{e^{20}}{20} - \frac{n}{20} \cdot I_{n-1} \Rightarrow 20I_n + nI_{n-1} = e^{20}$$

Put  $n = 10$  and  $n = 9$ , we get

$$20I_{10} + 10I_9 = e^{20} \dots (i)$$

$$\text{and } 20I_9 + 9I_8 = e^{20} \dots (ii)$$

From Eqs. (i) and (ii),

$$20I_{10} - 10I_9 - 9I_8 = 0$$

$$\Rightarrow 20I_{10} = 10I_9 + 9I_8 \text{ comparing this to}$$

$$20(I_{10}) = \alpha I_9 + \beta I_8, \text{ we get}$$

$$\alpha = 10, \beta = 9$$

$$\therefore \alpha - \beta = 1$$

---

## Question 135

Let  $f : (0, 2) \rightarrow \mathbb{R}$  be defined as  $f(x) = \log_2 \left[ 1 + \tan \left( \frac{\pi x}{4} \right) \right]$  Then,

$\lim_{n \rightarrow \infty} \frac{2}{n} \left[ f \left( \frac{1}{n} \right) + f \left( \frac{2}{n} \right) + \dots + f(1) \right]$  is equal to.....

[16 Mar 2021 Shift 1]

**Answer: 1**

**Solution:**

**Solution:**

$$f(x) = \log_2 \left[ 1 + \tan \left( \frac{\pi x}{4} \right) \right]$$

$$\text{Then, } = \lim_{n \rightarrow \infty} \frac{2}{n} \left[ f \left( \frac{1}{n} \right) + f \left( \frac{2}{n} \right) + \dots + f(1) \right] = 2 \lim_{n \rightarrow \infty} \sum_{r=1}^n \left( \frac{1}{n} \right) f \left( \frac{r}{n} \right)$$

$$\text{Let } I = \frac{2}{\log_n 2} \int_0^1 \log_n \left[ 1 + \tan \left( \frac{\pi x}{4} \right) \right] dx \dots (i)$$

$$\text{as, } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$\text{So, } x \rightarrow 1-x$$

$$I = \frac{2}{\log_n 2} \int_0^1 \log_n \left[ 1 + \tan \frac{\pi}{4} (1-x) \right] dx$$

$$= \frac{2}{\log_n 2} \int_0^1 \log_n \left[ 1 + \tan \left( \frac{\pi}{4} - \frac{\pi x}{4} \right) \right] dx$$



$$\begin{aligned}
&= \frac{2}{\log_n 2} \int_0^1 \log_n \left[ 1 + \left( \frac{1 - \tan \pi x/4}{1 + \tan \pi x/4} \right) \right] dx \\
&= \frac{2}{\log_n 2} \int_0^1 \log_n \left( \frac{2}{1 + \tan \frac{\pi x}{4}} \right) dx \\
&= \frac{2}{\log_n 2} \int_0^1 \log_n 2 - \log_n \left( 1 + \tan \frac{\pi x}{4} \right) dx \dots (ii)
\end{aligned}$$

Adding Eqs. (i) and (ii), we get

$$2I = \frac{2}{\log_n 2} \int_0^1 \log_n 2 dx$$

$$I = 1$$

## Question 136

If the normal to the curve  $y(x) = \int_0^x (2t^2 - 15t + 10) dt$  at a point  $(a, b)$  is parallel to the line  $x + 3y = -5$ ,  $a > 1$ , then the value of  $|a + 6b|$  is equal to.....  
[16 Mar 2021 Shift 1]

**Answer: 406**

### Solution:

**Solution:**

$$\text{Given, } y(x) = \int_0^x (2t^2 - 15t + 10) dt$$

$$\Rightarrow y'(x) = 2x^2 - 15x + 10$$

Since equation of normal is parallel to  $x + 3y = -5$

$\therefore$  Slope of normal to  $y(x) =$  Slope of line

$$\Rightarrow \frac{-1}{[y'(x)]_{a,b}} = \frac{-1}{3}$$

$$\text{or } [y'(x)]_{a,b} = 3$$

$$2a^2 - 15a + 10 = 3$$

$$\Rightarrow 2a^2 - 15a + 7 = 0$$

$$\Rightarrow (2a - 1)(a - 7) = 0$$

$$a = \frac{1}{2} \text{ or } 7$$

As,  $a > 1$ , so,  $a = 7$

Now,  $(7, b)$  lies on  $y(x)$ ,

$$\therefore b = \int_0^7 (2t^2 - 15t + 10) dt$$

$$\Rightarrow b = \frac{2}{3}a^3 - \frac{15}{2}a^2 + 10a$$

$$\Rightarrow b = \frac{2}{3}(7)^3 - \frac{15}{2}(7)^2 + 10(7)$$

$$\Rightarrow b = \frac{-413}{6}$$

$$\text{So, } a + 6b = 7 - 6\left(\frac{413}{6}\right) = -406$$

$$\therefore |a + 6b| = 406$$

## Question 137

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that  $f(x) + f(x+1) = 2$ , for all  $x \in \mathbb{R}$ . If  $I_1 = \int_0^8 f(x) dx$  and  $I_2 = \int_{-1}^3 f(x) dx$ , then the value of  $I_1 + 2I_2$  is equal to.....

[16 Mar 2021 Shift 1]

**Answer: 16**

**Solution:**

**Solution:**

Given,  $f(x) + f(x+1) = 2 \dots (i)$

$$I_1 = \int_0^8 f(x) dx$$

$$\text{and } I_2 = \int_{-1}^3 f(x) dx$$

Let  $f(0) = a$

Put  $x = 0$  in Eq. (i)

$$f(0) + f(1) = 2$$

$$f(1) = 2 - a$$

Put  $x = 1$  in Eq. (i)

$$f(1) + f(2) = 2$$

$f(2) = a$  and so on

So,  $f(0) = f(2) = f(4) \dots = a$

$f(1) = f(3) = f(5) \dots = 2 - a$

Clearly,  $f(x)$  is periodic with its period 2 units.

$$\text{So, } I_1 = \int_0^8 f(x) dx$$

$$\Rightarrow I_1 = 4 \int_0^2 f(x) dx$$

$$\text{Now, } I_2 = \int_{-1}^3 f(x) dx$$

$x \rightarrow x+1$

$$I_2 = \int_0^4 f(x+1) dx = \int_0^4 [2 - f(x)] dx$$



$$\begin{aligned} \Rightarrow I_2 &= 8 - 2 \int_0^2 f(x) dx \\ \Rightarrow 2I_2 &= 16 - 4 \int_0^2 f(x) dx \\ \Rightarrow 2I_2 &= 16 - I_1 \\ \therefore I_1 + 2I_2 &= 16 \end{aligned}$$


---

## Question 138

Consider the integral  $I = \int_0^{10} \frac{[x]e^{[x]}}{e^{x-1}} dx$  where  $[x]$  denotes the greatest integer less than or equal to  $x$ . Then, the value of  $I$  is equal to  
[16 Mar 2021 Shift 2]

Options:

- A.  $9(e - 1)$
- B.  $45(e + 1)$
- C.  $45(e - 1)$
- D.  $9(e + 1)$

Answer: C

Solution:

Solution:

We have,  $\int_0^{10} \frac{[x]e^{[x]}}{e^{x-1}} dx$

$$= e \int_0^{10} \frac{[x]e^{[x]}}{e^x} dx$$

$$= e \int_0^1 \frac{0}{e^x} dx + e \int_1^2 \frac{e}{e^x} dx + e \int_2^3 \frac{2e^2}{e^x} dx + \dots$$

$$\Rightarrow \int_a^b e^{-x} dx = \frac{e^{-x}}{-1} \Big|_a^b \Rightarrow (e^{-a} - e^{-b})$$

$$\Rightarrow e^2 \left( \frac{1}{e} - \frac{1}{e^2} \right) + 2e^3 \left( \frac{1}{e^2} - \frac{1}{e^3} \right) + 3e^4 \left( \frac{1}{e^3} - \frac{1}{e^4} \right) + \dots + 9e^{10} \left( \frac{1}{e^9} - \frac{1}{e^{10}} \right)$$

$$= (e - 1) + 2(e - 1) + 3(e - 1) + \dots + 9(e - 1)$$

$$= (1 + 2 + 3 + \dots + 9)(e - 1) = \left( \frac{9 \times 10}{2} \right) (e - 1)$$

$$= 45(e - 1)$$


---



## Question139

Let the domain of the function  $f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$  be

(a, b) Then the value of the integral  $\int_a^b \frac{\sin^3 x}{(\sin^3 x + \sin^3(a+b-x))} dx$  is equal to \_\_\_\_\_.

[27 Jul 2021 Shift 1]

Answer: 1

Solution:

Solution:

For domain

$$\log_5(\log_3(18x - x^2 - 77)) > 0$$

$$\log_3(18x - x^2 - 77) > 1$$

$$18x - x^2 - 77 > 3$$

$$x^2 - 18x + 80 < 0$$

$$x \in (8, 10)$$

$$\Rightarrow a = 8 \text{ and } b = 10$$

$$I = \int_a^b \frac{\sin^3 x}{\sin^3 x + \sin^3(a+b-x)} dx$$

$$I = \int_a^b \frac{\sin^3(a+b-x)}{\sin^3 x + \sin^3(a+b-x)} dx$$

$$2I = (b-a) \Rightarrow I = \frac{b-a}{2} (\because a = 8 \text{ and } b = 10)$$

$$I = \frac{10-8}{2} = 1$$

---

## Question140

The value of  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{(2j-1)+8n}{(2j-1)+4n}$  is equal to :

[27 Jul 2021 Shift 1]

Options:

A.  $5 + \log_e\left(\frac{3}{2}\right)$

B.  $2 - \log_e\left(\frac{2}{3}\right)$

C.  $3 + 2\log_e\left(\frac{2}{3}\right)$

D.  $1 + 2\log_e\left(\frac{3}{2}\right)$

**Answer: D**

**Solution:**

**Solution:**

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{\left(\frac{2j}{n} - \frac{1}{n} + 8\right)}{\left(\frac{2j}{n} - \frac{1}{n} + 4\right)}$$
$$\int_0^1 \frac{2x+8}{2x+4} dx = \int_0^1 dx + \int_0^1 \frac{4}{2x+4} dx$$
$$= 1 + 4 \frac{1}{2} (\ln |2x+4|) \Big|_0^1$$
$$= 1 + 2 \ln\left(\frac{3}{2}\right)$$

---

## Question 141

The value of the definite integral

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1 + e^{x \cos x})(\sin^4 x + \cos^4 x)}$$
 is equal to:

[27 Jul 2021 Shift 1]

**Options:**

A.  $-\frac{\pi}{2}$

B.  $\frac{\pi}{2\sqrt{2}}$

C.  $-\frac{\pi}{4}$

D.  $\frac{\pi}{\sqrt{2}}$

**Answer: B**

**Solution:**

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1 + e^{x \cos x})(\sin^4 x + \cos^4 x)} \dots\dots\dots(1)$$

Using  $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1 + e^{-x \cos x})(\sin^4 x + \cos^4 x)} \dots\dots\dots(2)$$

Add (1) and (2)

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x + \cos^4 x}$$

$$2I = 2 \int_0^{\frac{\pi}{4}} \frac{dx}{\sin^4 x + \cos^4 x}$$

$$I = \int_0^{\frac{\pi}{4}} \frac{(1 + \tan^2 x) \sec^2 x}{\tan^4 x + 1} dx$$

$$I = \int_0^{\frac{\pi}{4}} \left(1 + \frac{1}{\tan^2 x}\right) \sec^2 x \frac{x}{\left(\tan x - \frac{1}{\tan x}\right)^2 + 2} dx$$

$$\tan x - \frac{1}{\tan x} = t$$

$$\left(1 + \frac{1}{\tan^2 x}\right) \sec^2 x dx = dt$$

$$I = \int_{-\infty}^0 \frac{dt}{t^2 + 2} = \left[ \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{t}{\sqrt{2}} \right) \right]_{-\infty}^0$$

$$I = 0 - \frac{1}{\sqrt{2}} \left( -\frac{\pi}{2} \right) = \frac{\pi}{2\sqrt{2}}$$

## Question 142

The value of the definite integral  $\int_{\pi/24}^{5\pi/24} \frac{dx}{1 + \sqrt[3]{\tan 2x}}$  is

[25 Jul 2021 Shift 1]

Options:

A.  $\frac{\pi}{3}$

B.  $\frac{\pi}{6}$

C.  $\frac{\pi}{12}$

D.  $\frac{\pi}{18}$

**Answer: C**

**Solution:**

**Solution:**

$$\text{Let } I = \int_{\pi/24}^{5\pi/24} \frac{(\cos 2x)^{1/3}}{(\cos 2x)^{1/3} + (\sin 2x)^{1/3}} dx \dots\dots(i)$$

$$\Rightarrow I = \int_{\pi/24}^{5\pi/24} \frac{\left(\cos \left\{ 2\left(\frac{\pi}{4} - x\right) \right\}\right)^{\frac{1}{3}}}{\left(\cos \left\{ 2\left(\frac{\pi}{4} - x\right) \right\}\right)^{\frac{1}{3}} + \left(\sin \left\{ 2\left(\frac{\pi}{4} - x\right) \right\}\right)^{\frac{1}{3}}} dx$$

$$\left\{ \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right\}$$

$$\text{So } I = \int_{\pi/24}^{5\pi/24} \frac{(\sin 2x)^{1/3}}{(\sin 2x)^{1/3} + (\cos 2x)^{1/3}} dx \dots\dots(ii)$$

$$\text{Hence } 2I = \int_{\pi/24}^{5\pi/24} dx$$

[(i) + (ii)]

$$\Rightarrow 2I = \frac{4\pi}{24} \Rightarrow I = \frac{\pi}{12}$$

## Question 143

The value of the integral  $\int_{-1}^1 \log(x + \sqrt{x^2 + 1}) dx$  is:  
[25 Jul 2021 Shift 2]

**Options:**

A. 2

B. 0

C. -1

D. 1

**Answer: B**

**Solution:**

**Solution:**

$$\text{Let } I = \int_{-1}^1 \log(x + \sqrt{x^2 + 1}) dx$$

$\therefore \log(x + \sqrt{x^2 + 1})$  is an odd function

$$\therefore I = 0$$

## Question 144

If  $\int_0^{100\pi} \frac{\sin^2 x}{e^{\left(\frac{x}{\pi} \left[ \frac{x}{\pi} \right] \right)}} dx = \frac{\alpha \pi^3}{1 + 4\pi^2}$ ,  $\alpha \in \mathbf{R}$  where  $[x]$  is the greatest integer less than or equal

to  $x$ , then the value of  $\alpha$  is :  
[22 Jul 2021 Shift 2]

**Options:**

A.  $200(1 - e^{-1})$

B.  $100(1 - e)$

C.  $50(e - 1)$

D.  $150(e^{-1} - 1)$

**Answer: A**

**Solution:**

**Solution:**

$$I = \int_0^{100\pi} \frac{\sin^2 x}{e^{\left(\frac{x}{\pi} \left[ \frac{x}{\pi} \right] \right)}} dx = 100 \int_0^{\pi} \frac{\sin^2 x}{e^{x/\pi}} dx$$

$$100 \int_0^{\pi} e^{-x/\pi} \frac{(1 - \cos 2x)}{2} dx$$

$$= 50 \left\{ \int_0^{\pi} e^{-x/\pi} dx - \int_0^{\pi} e^{-x/\pi} \cos 2x dx \right\}$$

$$I_1 = \int_0^{\pi} e^{-x/\pi} dx = [-\pi e^{-x/\pi}]_0^{\pi} = \pi(1 - e^{-1})$$

$$I_2 = \int_0^{\pi} e^{-x/\pi} \cos 2x dx$$

$$= -\pi e^{-x/\pi} \cos 2x \Big|_0^{\pi} - \int_0^{\pi} -\pi e^{-x/\pi} (-2 \sin 2x) dx$$

$$= \pi(1 - e^{-1}) - 2\pi \int_0^{\pi} e^{-x/\pi} \sin 2x dx$$

$$= \pi(1 - e^{-1}) - 2\pi \left\{ -\pi e^{-x/\pi} \sin 2x \Big|_0^{\pi} - \int_0^{\pi} -\pi e^{-x/\pi} 2 \cos 2x dx \right\} = \pi(1 - e^{-1}) - 4\pi^2 I_2$$

$$\Rightarrow I_2 = \frac{\pi(1 - e^{-1})}{1 + 4\pi^2}$$



$$\begin{aligned} \therefore I &= 50 \left\{ \pi(1 - e^{-1}) - \frac{\pi(1 - e^{-1})}{1 + 4\pi^2} \right\} \\ &= \frac{200(1 - e^{-1})\pi^3}{1 + 4\pi^2} \end{aligned}$$


---

## Question 145

The value of the integral  $\int_{-1}^1 \log_e(\sqrt{1-x} + \sqrt{1+x}) dx$  is equal to :  
[20 Jul 2021 Shift 1]

Options:

A.  $\frac{1}{2} \log_e 2 + \frac{\pi}{4} - \frac{3}{2}$

B.  $2 \log_e 2 + \frac{\pi}{4} - 1$

C.  $\log_e 2 + \frac{\pi}{2} - 1$

D.  $2 \log_e 2 + \frac{\pi}{2} - \frac{1}{2}$

Answer: B

Solution:

Solution:

$$\text{Let } I = 2 \int_0^1 \ln(\sqrt{1-x} + \sqrt{1+x}) dx$$

$$\therefore I = 2 \left[ (x \cdot \ln(\sqrt{1-x} + \sqrt{1+x}))_0^1 - \int_0^1 x \cdot \left( \frac{1}{\sqrt{1-x} + \sqrt{1+x}} \right) \cdot \left( \frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}} \right) dx \right]$$

$$= 2(\ln \sqrt{2} - 0) - \frac{2}{2} \int_0^1 \frac{x\sqrt{1-x} - \sqrt{1+x} dx}{(\sqrt{1-x} + \sqrt{1+x})\sqrt{1-x^2}}$$

$$= (\log_e 2) - \int_0^1 \frac{x \cdot (2 - 2\sqrt{1-x^2})}{-2x\sqrt{1-x^2}} dx \text{ (After rationalisation)}$$

$$= (\log_e 2) + \int_0^1 \left( \frac{1 - \sqrt{1-x^2}}{\sqrt{1-x^2}} \right) dx$$

$$= (\log_e 2) + (\sin^{-1} x)_0^1 - 1$$

$$= \log_e 2 + \left( \frac{\pi}{2} - 0 \right) - 1$$

$$\therefore I = (\log_e 2) + \frac{\pi}{2} - 1$$

⇒ Option (3) is correct.

## Question 146

Let  $a$  be a positive real number such that  $\int_0^a e^{x - [x]} dx = 10e - 9$  where  $[x]$  is the greatest integer less than or equal to  $x$ . Then  $a$  is equal to :

[20 Jul 2021 Shift 1]

Options:

A.  $10 - \log_e(1 + e)$

B.  $10 + \log_e 2$

C.  $10 + \log_e 3$

D.  $10 + \log_e(1 + e)$

**Answer: B**

**Solution:**

**Solution:**

$$a > 0$$

$$\text{Let } n \leq a < n + 1, n \in \mathbb{W}$$

$$\therefore a = \underbrace{[a]}_{\downarrow} + \underbrace{\{a\}}_{\downarrow}$$

G . I . F Fractional part

$$\text{Here } [a] = n \text{ Now, } \int_0^a e^{x - [x]} dx = 10e - 9$$

$$\Rightarrow \int_0^n e^{\{x\}} dx + \int_n^a e^{x - [x]} dx = 10e - 9$$

$$\therefore n \int_0^1 e^x dx + \int_n^a e^{x - n} dx = 10e - 9$$

$$\Rightarrow n(e - 1) + (e^{a - n} - 1) = 10e - 9$$

$$\therefore n = 0 \text{ and } \{a\} = \log_e 2$$

$$\text{So, } a = [a] + \{a\} = (10 + \log_e 2)$$

$\Rightarrow$  Option (2) is correct.

---

## Question 147

If  $[x]$  denotes the greatest integer less than or equal to  $x$ , then the value of the integral  $\int_{-\pi/2}^{\pi/2} [[x] - \sin x] dx$  is equal to :

[20 Jul 2021 Shift 2]

Options:

A.  $-\pi$

B.  $\pi$

C. 0

D. 1

Answer: A

Solution:

Solution:

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ([x] + [-\sin x]) dx \dots\dots(1)$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ([-x] + [\sin x]) dx \dots\dots(2)$$

(King property)

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ([x] + [-x]) + ([\sin x] + [-\sin x]) dx$$

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-2) dx = -2(\pi)$$

$$I = -\pi$$

---

## Question 148

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f(x) = x + 1$ , then the value of

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[ f(0) + f\left(\frac{5}{n}\right) + f\left(\frac{10}{n}\right) + \dots\dots\dots + f\left(\frac{5(n-1)}{n}\right) \right] \text{ is:}$$

[20 Jul 2021 Shift 2]

Options:

A.  $\frac{3}{2}$

B.  $\frac{5}{2}$

C.  $\frac{1}{2}$

D.  $\frac{7}{2}$

**Answer: D**

**Solution:**

**Solution:**

$$I = \sum_{r=0}^{n-1} f\left(\frac{5r}{n}\right) \frac{1}{n}$$

$$I = \int_0^1 f(5x) dx$$

$$I = \int_0^1 (5x+1) dx$$

$$I = \left[ \frac{5x^2}{2} + x \right]_0^1$$

$$I = \frac{5}{2} + 1 = \frac{7}{2}$$

---

## Question149

Let  $g(t) = \int_{-\pi/2}^{\pi/2} \cos\left(\frac{\pi}{4}t + f(x)\right) dx$ , where  $f(x) = \log_e\left(x + \sqrt{x^2 + 1}\right)$ ,  $x \in \mathbf{R}$ . Then which one of the following is correct?  
[20 Jul 2021 Shift 2]

**Options:**

A.  $g(1) = g(0)$

B.  $\sqrt{2}g(1) = g(0)$

C.  $g(1) = \sqrt{2}g(0)$

D.  $g(1) + g(0) = 0$

**Answer: B**

**Solution:**

$$g(t) = \int_{-\pi/2}^{\pi/2} \left( \cos \frac{\pi}{4} t + f(x) \right) dx$$

$$g(t) = \pi \cos \frac{\pi}{4} t + \int_{-\pi/2}^{\pi/2} f(x) dx$$

$$g(t) = \pi \cos \frac{\pi}{4} t$$

$$g(1) = \frac{\pi}{\sqrt{2}}, g(0) = \pi$$

## Question 150

Let  $F : [3, 5] \rightarrow \mathbb{R}$  be a twice differentiable function on  $(3, 5)$  such that

$$F(x) = e^{-x} \int_3^x (3t^2 + 2t + 4F'(t)) dt$$

If  $F'(4) = \frac{\alpha e^\beta - 224}{(e^\beta - 4)^2}$ , then  $\alpha + \beta$  is equal to \_\_\_\_\_.

[27 Jul 2021 Shift 1]

**Answer: 16**

**Solution:**

**Solution:**

$$F(3) = 0$$

$$e^x F(x) = \int_3^x (3t^2 + 2t + 4F'(t)) dt$$

$$e^x F(x) + e^x F'(x) = 3x^2 + 2x + 4F'(x)$$

$$(e^x - 4) \frac{dy}{dx} + e^x y = (3x^2 + 2x)$$

$$\frac{dy}{dx} + \frac{e^x}{(e^x - 4)} y = \frac{(3x^2 + 2x)}{(e^x - 4)}$$

$$y e^{\int \frac{e^x}{e^x - 4} dx} = \int \frac{(3x^2 + 2x)}{(e^x - 4)} e^{\int \frac{e^x}{e^x - 4} dx} dx$$

$$y \cdot (e^x - 4) = \int (3x^2 + 2x) dx + c$$

$$y(e^x - 4) = x^3 + x^2 + c$$

$$\text{Put } x = 3 \Rightarrow c = -36$$

$$F(x) = \frac{(x^3 + x^2 - 36)}{(e^x - 4)}$$

$$F'(x) = \frac{(3x^2 + 2x)(e^x - 4) - (x^3 + x^2 - 36)e^x}{(e^x - 4)^2}$$

Now put value of  $x = 4$  we will get  $\alpha = 12$  &  $\beta = 4$

## Question151

If  $\int_0^{\pi} (\sin^3 x) e^{-\sin^2 x} dx = \alpha - \frac{\beta}{e} \int_0^1 \sqrt{t} e^t dt$ , then  $\alpha + \beta$  is equal to \_\_\_\_\_ .

[27 Jul 2021 Shift 2]

Answer: 5

Solution:

Solution:

$$\begin{aligned} I &= \int_0^{\pi} (\sin^3 x) e^{-\sin^2 x} dx \\ &= 2 \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx + \int_0^{\pi/2} \cos x e^{-\sin^2 x} (-\sin 2x) dx \\ &= 2 \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx + [\cos x e^{-\sin^2 x}]_0^{\pi/2} + \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx \\ &= 3 \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx - 1 \\ &= \frac{3}{2} \int_{-1}^0 \frac{e^{\alpha} d\alpha}{\sqrt{1+\alpha}} - 1 \quad (\text{Put } -\sin^2 x = t) \\ &= \frac{3}{2e} \int_0^1 \frac{e^x}{\sqrt{x}} dx - 1 \quad (\text{put } 1 + \alpha = x) \\ &= \frac{3}{2e} \int_0^1 e^x \frac{1}{\sqrt{x}} dx - 1 \\ &= 2 - \frac{3}{e} \int_0^1 e^x \sqrt{x} dx \end{aligned}$$

Hence,  $\alpha + \beta = 5$

---

## Question152

The value of  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{1 + \sin^2 x}{1 + \pi^{\sin x}} \right) dx$  is

[26 Aug 2021 Shift 2]

Options:

A.  $\frac{\pi}{2}$

B.  $\frac{5\pi}{2}$

C.  $\frac{3\pi}{4}$

D.  $\frac{3\pi}{2}$

**Answer: C**

**Solution:**

**Solution:**

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \sin^2 x}{1 + \pi^{\sin x}} dx \dots (i)$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \sin^2(-x)}{1 + \pi^{\sin(-x)}} dx \left[ \because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \sin x}{1 + \pi^{-\sin x}} dx$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi^{\sin x} (1 + \sin^2 x)}{1 + \pi^{\sin x}} dx \dots (ii)$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1 + \pi^{\sin x})(1 + \sin^2 x)}{(1 + \pi^{\sin x})} dx$$

$$\Rightarrow 2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin^2 x) dx$$

$$\Rightarrow 2I = [x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} \sin^2 x dx$$

$$[\because \sin^2 x \text{ is an even function, so } \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 dx = 2 \int_0^{\frac{\pi}{2}} \sin^2 x dx]$$

$$\Rightarrow I = \frac{1}{2} \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right) + \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2x) dx$$

$$\Rightarrow I = \frac{\pi}{2} + \frac{1}{2} \left[ x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}}$$

$$\frac{\pi}{2} + \frac{1}{2} \left[ \left( \frac{\pi}{2} - 0 \right) - (0 - 0) \right]$$

$$I = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$


---

## Question 153

If  $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = a \sin^{-1} \left( \frac{\sin x + \cos x}{b} \right) + c$ , where  $c$  is a constant of integration, then the ordered pair  $(a, b)$  is equal to [2021]

**Options:**

- A.  $(-1, 3)$
- B.  $(3, 1)$
- C.  $(1, 3)$
- D.  $(1, -3)$

**Answer: C**

**Solution:**

**Solution:**

$$\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx$$

$$\int \frac{\cos x - \sin x}{\sqrt{9 - (\sin x + \cos x)^2}} dx$$

Let  $\sin x + \cos x = t$

$$\int \frac{dt}{\sqrt{9 - t^2}} = \sin^{-1} \frac{t}{3} + c$$

$$= \sin^{-1} \left( \frac{\sin x + \cos x}{3} \right) + c$$

So  $a = 1, b = 3$ .

---

## Question 154



The integral  $\int \frac{1}{\sqrt[4]{(x-1)^3(x+2)^5}} dx$  is equal to (where C is a constant of integration)

[31 Aug 2021 Shift 1]

Options:

A.  $\frac{3}{4} \left( \frac{x+2}{x-1} \right)^{\frac{1}{4}} + C$

B.  $\frac{3}{4} \left( \frac{x+2}{x-1} \right)^{\frac{5}{4}} + C$

C.  $\frac{4}{3} \left( \frac{x-1}{x+2} \right)^{\frac{1}{4}} + C$

D.  $\frac{4}{3} \left( \frac{x-1}{x+2} \right)^{\frac{5}{4}} + C$

Answer: C

Solution:

Solution:

$$\int \frac{1}{\frac{3}{(x-1)^4} \frac{5}{(x+2)^4}} dx = \int \frac{dx}{\left( \frac{x+2}{x-1} \right)^{\frac{5}{4}} (x-1)^2}$$

$$\frac{x+2}{x-1} = t$$

$$\Rightarrow \left( \frac{(x-1) - (x+2)}{(x-1)^2} \right) dx = dt$$

$$\Rightarrow -\frac{3}{(x-1)^2} dx = dt$$

$$\Rightarrow -\frac{1}{3} \int \frac{dt}{t^{5/4}} = \frac{4}{3} \cdot \frac{1}{t^{1/4}} + C$$

$$= \frac{4}{3} \left( \frac{x-1}{x+2} \right)^{\frac{1}{4}} + C$$

---

## Question 155

If

$$\int \frac{\sin x}{\sin^3 x + \cos^3 x} dx = \alpha \log_e \left| 1 + \tan x \right| + \beta \log_e \left| 1 - \tan x + \tan^2 x \right| + \gamma \tan^{-1} \left( \frac{2 \tan x - 1}{\sqrt{3}} \right) + C,$$

when  $C$  is constant of integration, then the value of  $18(\alpha + \beta + \gamma^2)$  is  
**[31 Aug 2021 Shift 2]**

**Answer: 3**

**Solution:**

**Solution:**

$$\text{Let } I = \int \frac{\sin x}{\sin^3 x + \cos^3 x} dx =$$

$$= \frac{\tan x \sec^2 x}{\tan^3 x + 1} dx$$

$$\text{Put } \tan x = t$$

$$\Rightarrow \sec^2 x dx = dt$$

$$I = \int \frac{tdt}{t^3 + 1} = \int \frac{t}{(t+1)(t^2 - t + 1)} dt$$

$$\text{Now, } \frac{t}{(t+1)(t^2 - t + 1)} = \frac{A}{t+1} + Bt + Ct^2 - t + 1$$

$$\Rightarrow t = A(t^2 - t + 1) + (Bt + C)(t + 1)$$

Comparing coefficients to both the sides and solving them for  $A, B, C$ , we have

$$A = -\frac{1}{3}, B = \frac{1}{3}, C = \frac{1}{3}$$

$$\text{Hence, } I = -\frac{1}{3} \int \frac{1}{t+1} dt + \frac{1}{3} \int \frac{t+1}{t^2 - t + 1} dt$$

$$= -\frac{1}{3} \ln(t+1) + \frac{1}{3} \int \frac{\frac{1}{2}(2t-1) + \frac{3}{2}}{t^2 - t + 1} dt$$

$$= -\frac{1}{3} \ln(t+1) + \frac{1}{6} \ln(t^2 - t + 1) + \frac{1}{2} \int \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3} \ln(t+1) + \frac{1}{6} \ln(t^2 - t + 1) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2t-1}{\sqrt{3}} \right) + C$$

$$= -\frac{1}{3} \ln(\tan x + 1) + \frac{1}{6} \ln(\tan^2 x - \tan x + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2 \tan x - 1}{\sqrt{3}} \right) + C$$

$$\Rightarrow \alpha = \frac{-1}{3}, \beta = \frac{1}{6}, \gamma = \frac{1}{\sqrt{3}}$$

$$\text{So, } 18(\alpha + \beta + \gamma^2) = 18 \left( \frac{-1}{3} + \frac{1}{6} + \frac{1}{3} \right) = 3$$

## Question 156

If  $\int \frac{dx}{(x^2+x+1)^2} = a \tan^{-1} \left( \frac{2x+1}{\sqrt{3}} \right) + b \left( \frac{2x+1}{x^2+x+1} \right) + C$ ,  $x > 0$  where  $C$  is the constant of integration, then the value of  $9(\sqrt{3}a + b)$  is equal to  
**[27 Aug 2021 Shift 1]**

**Answer: 15**

**Solution:**

**Solution:**

$$\int \frac{dx}{(x^2+x+1)^2} = \int \frac{dx}{\left[ \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \right]^2}$$

$$\text{Let } x + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta$$

$$\Rightarrow dx = \frac{\sqrt{3}}{2} \sec^2 \theta d\theta$$

$$\therefore \int \frac{\frac{\sqrt{3}}{2} \sec^2 \theta d\theta}{\frac{9}{16} (\tan^2 \theta + 1)^2} = \frac{8}{3\sqrt{3}} \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta}$$

$$= \frac{8}{3\sqrt{3}} \int \cos^2 \theta d\theta$$

$$= \frac{8}{3\sqrt{3}} \int \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \frac{4}{3\sqrt{3}} (\theta + \sin 2\theta) + C$$

$$= \frac{4}{2\sqrt{3}} \tan^{-1} \left( \frac{2x+1}{\sqrt{3}} \right) + \frac{4}{3\sqrt{3}} \frac{\frac{2x+1}{\sqrt{3}}}{1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2} + C$$

$$= \frac{4}{3\sqrt{3}} \tan^{-1} \left( \frac{2x+1}{\sqrt{3}} \right) + \frac{1}{3} \frac{2x+1}{x^2+x+1} + C$$

$$\therefore a = \frac{4}{3\sqrt{3}}, b = \frac{1}{3}$$

$$\text{Hence, } 9(\sqrt{3}a + b) = 9\left(\frac{4}{3} + \frac{1}{3}\right) = 15$$

## Question 157

$$\text{If } \int \frac{2e^x + 3e^{-x}}{4e^x + 7e^{-x}} dx = \frac{1}{14} (ux + v \log_e (4e^x + 7e^{-x})) + C,$$

where  $C$  is a constant of integration, then  $u + v$  is equal to  
[27 Aug 2021 Shift 2]

Answer: 7

Solution:

Solution:

$$I = \int \frac{2e^x + 3e^{-x}}{4e^x + 7e^{-x}} dx = \int \frac{2e^{2x} + 3}{4e^{2x} + 7} dx$$

$$\text{Let } 2e^{2x} + 3 = A \frac{d}{dx}(4e^{2x} + 7) + B(4e^{2x} + 7)$$

$$\Rightarrow 2e^{2x} + 3 = (8A + 4B)e^{2x} + 7B$$

Comparing both sides

$$B = \frac{3}{7} \text{ and } A = \frac{1}{28}$$

$$\therefore I = \int \frac{\frac{1}{28}(8e^{2x}) + \frac{3}{7}(4e^{2x} + 7)}{4e^{2x} + 7} dx$$

$$= \frac{1}{28} \ln |4e^{2x} + 7| + \frac{3}{7}x + C$$

$$= \frac{1}{28} \ln |e^x(4e^x + 7e^{-x})| + \frac{3}{7}x + C$$

$$= \frac{1}{28}x + \frac{1}{28} \ln |4e^x + 7e^{-x}| + \frac{3}{7}x + C$$

$$= \frac{1}{14} \left( \frac{13}{2}x + \frac{1}{2} \ln |4e^x + 7e^{-x}| \right) + C$$

$$\Rightarrow u = \frac{13}{2} \text{ and } v = \frac{1}{2}$$

$$\therefore u + v = \frac{13}{2} + \frac{1}{2} = 7$$

---

## Question 158

Section B : Numerical Type Questions

Let  $[t]$  denote the greatest integer  $\leq t$ . Then the value of  $8 \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} ([2x] + |x|) dx$  is

[31 Aug 2021 Shift 1]

**Answer: 5**

**Solution:**

**Solution:**

$$8 \int_{-\frac{1}{2}}^1 ([2x] + |x|) dx$$

$$= -\frac{1}{2} \leq x < 0$$

$$\Rightarrow [2x] = -1$$

$$0 \leq x < \frac{1}{2}$$

$$\Rightarrow [2x] = 0$$

$$\frac{1}{2} \leq x < 1$$

$$\Rightarrow [2x] = 1$$

$$I = \int_{-\frac{1}{2}}^0 (-1 - x) dx + \int_0^{\frac{1}{2}} (0 + x) dx + \int_{\frac{1}{2}}^1 (1 + x) dx$$

$$= \left[ -x - \frac{x^2}{2} \right]_{-\frac{1}{2}}^0 + \left[ \frac{x^2}{2} \right]_0^{\frac{1}{2}} + \left[ x + \frac{x^2}{2} \right]_{\frac{1}{2}}^1$$

$$= -\left(\frac{1}{2} - \frac{1}{8}\right) + \left(\frac{1}{8}\right) + \left(1 + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{8}\right) = \frac{5}{8}$$

$$\therefore 8I = 8 \cdot \frac{5}{8} = 5$$

---

## Question 159

If  $x\phi(x) = \int_5^x (3t^2 - 2\phi'(t)) dt$ ,  $x > -2$ , and  $\phi(0) = 4$ , then  $\phi(2)$  is  
[31 Aug 2021 Shift 1]

**Answer: 4**

**Solution:**

**Solution:**

$$x\phi(x) = \int_5^x (3t^2 - 2\phi'(t)) dt$$

$$\Rightarrow x\phi(x) = [t^3 - 2\phi(t)]_5^x$$

$$\Rightarrow x\phi(x) = (x^3 - 125) - 2[\phi(x) - \phi(5)]$$

$$\text{Now, } \phi(0) = 4$$

$$\Rightarrow 0 = -125 - 2[4 - \phi(5)]$$

$$\Rightarrow \phi(5) = \frac{133}{2}$$

For  $\phi(2)$ ,

$$\Rightarrow 2\phi(2) = (8 - 125) - 2\left[\phi(2) - \frac{133}{2}\right]$$

$$\Rightarrow 4\phi(2) = 16$$

$$\Rightarrow \phi(2) = 4$$

## Question 160

If  $[x]$  is the greatest integer  $\leq x$ , then  $\pi^2 \int_0^2 \left(\sin \frac{\pi x}{2}\right) (x - [x])^{[x]} dx$  is equal to  
**[31 Aug 2021 Shift 2]**

**Options:**

A.  $2(\pi - 1)$

B.  $4(\pi - 1)$

C.  $4(\pi + 1)$

D.  $2(\pi + 1)$

**Answer: B**

**Solution:**

**Solution:**

$$\begin{aligned} & \pi^2 \int_0^2 \sin\left(\frac{\pi x}{2}\right) (x - [x])^{[x]} dx \\ &= \pi^2 \int_0^1 \sin\left(\frac{\pi x}{2}\right) x^0 dx + \pi^2 \int_1^2 \sin\left(\frac{\pi x}{2}\right) (x - 1) dx \\ &= \pi^2 \left[ \frac{-2}{\pi} \cos \frac{\pi x}{2} \right]_0^1 + \pi^2 \left[ (x - 1) \frac{2}{\pi} \left( -\cos \frac{\pi x}{2} \right) \right]_1^2 + \pi^2 \int_1^2 \frac{2}{\pi} \cos \frac{\pi x}{2} dx \\ &= \pi^2 \left( \frac{2}{\pi} \right) + \frac{2\pi^2}{\pi} (1 - 0) + 2\pi \cdot \frac{2}{\pi} \left( \sin \frac{\pi x}{2} \right) \Big|_1^2 \\ &= 2\pi + 2\pi + 4(0 - 1) = 4\pi - 4 = 4(\pi - 1) \end{aligned}$$

## Question 161

If  $U_n = \left(1 + \frac{1}{n^2}\right) \left(1 + \frac{2^2}{n^2}\right)^2 \dots \left(1 + \frac{n^2}{n^2}\right)^n$ , then  $\lim_{n \rightarrow \infty} (U_n)^{\frac{-4}{n^2}}$  is equal to

[27 Aug 2021 Shift 1]

Options:

A.  $\frac{e^2}{16}$

B.  $\frac{4}{e}$

C.  $\frac{16}{e^2}$

D.  $\frac{4}{e^2}$

Answer: A

Solution:

Solution:

$$\text{Let } y = \lim_{n \rightarrow \infty} (U_n)^{\frac{-4}{n^2}}$$

$$y = \lim_{n \rightarrow \infty} \left[ \left(1 + \frac{1}{n^2}\right)^{\frac{-4}{n^2}} \left(1 + \frac{2^2}{n^2}\right)^{\frac{-4}{n^2} \cdot 2} \left(1 + \frac{3^2}{n^2}\right)^{\frac{-4}{n^2} \cdot 3} \dots \right]$$

Taking log on both sides, we get

$$\ln y = \lim_{n \rightarrow \infty} \sum_{r=1}^n \left[ \frac{-4}{n^2} \cdot r \ln \left(1 + \frac{r^2}{n^2}\right) \right]$$

Now, replace  $\lim_{n \rightarrow \infty} \sum \rightarrow \int$

$$\frac{r}{n} \rightarrow x, \frac{1}{n} \rightarrow dx$$

Lower limit = 0

Upper limit = 1

$$\therefore \ln y = \int_0^1 -4x \ln(1+x^2) dx$$

Let  $1+x^2 = t$

$$\Rightarrow x dx = \frac{dt}{2}$$

When  $x \rightarrow 0$ ,  $t \rightarrow 1$

and  $x \rightarrow 1$ ,  $t \rightarrow 2$

$$\therefore \ln y = \int_1^2 -2 \ln t dt$$

$$= -2(t \ln t - t)_1^2$$

$$= -2(2 \ln 2 - 2 + 1)$$

$$= -2(2 \ln 2 - 1)$$

$$\Rightarrow \ln y = \ln \frac{1}{16} + 2$$

$$\Rightarrow y = \frac{1}{16} e^2$$

---

## Question 162

$\int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e (x^2 - 44x + 484)} dx$  is equal to

[27 Aug 2021 Shift 1]

Options:

A. 6

B. 8

C. 5

D. 10

Answer: C

Solution:

Solution:

$$\text{Let } I = \int_6^{16} \frac{\ln_e(x^2)}{\ln_e(x^2) + \ln_e(484 - 44x + x^2)} dx$$

$$= \int_6^{16} \frac{\ln_e(x^2)}{\ln_e(x^2) + \ln_e(22 - x)^2} dx$$

$$= \int_6^{16} \frac{2 \ln_e x dx}{2 \ln_e x + 2 \ln_e(22 - x)}$$

$$I = \int_6^{16} \frac{\ln_e x dx}{\ln_e x + \ln_e(22 - x)} \dots (i)$$

$$\because \int_a^b f(x) dx = \int_a^b f(a + b - x) dx$$

$$\therefore I = \int_6^{16} \frac{\ln_e(22 - x)}{\ln_e(22 - x) + \ln_e x} dx \dots (ii)$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_6^{16} \frac{\ln_e x + \ln_e(22 - x)}{\ln_e x + \ln_e(22 - x)} dx$$

$$2I = \int_6^{16} dx = x \Big|_6^{16} = 10$$

$$\text{or } I = 5$$

---

## Question 163





The value of the integral  $\int_0^1 \frac{\sqrt{x} dx}{(1+x)(1+3x)(3+x)}$  is  
**[27 Aug 2021 Shift 2]**

**Options:**

A.  $\frac{\pi}{8} \left( 1 - \frac{\sqrt{3}}{2} \right)$

B.  $\frac{\pi}{4} \left( 1 - \frac{\sqrt{3}}{6} \right)$

C.  $\frac{\pi}{8} \left( 1 - \frac{\sqrt{3}}{6} \right)$

D.  $\frac{\pi}{4} \left( 1 - \frac{\sqrt{3}}{2} \right)$

**Answer: A**

**Solution:**

**Solution:**

$$\int_0^1 \frac{\sqrt{x} dx}{(1+x)(1+3x)(3+x)}$$

Put  $\sqrt{x} = t$

$$\Rightarrow x = t^2$$

$$\text{or } dx = 2t dt$$

$$\therefore I = \int_0^1 \frac{2t^2 dt}{(t^2+1)(3t^2+1)(t^2+3)}$$

$$= \int_0^1 \frac{(3t^2+1) - (t^2+1)}{(t^2+1)(3t^2+1)(t^2+3)} dt$$

$$= \int_0^1 \left[ \frac{1}{(t^2+3)(t^2+1)} - \frac{1}{(t^2+3)(3t^2+1)} \right] dt$$

$$= \int_0^1 \left[ \frac{1}{2(t^2+1)} - \frac{1}{2(t^2+3)} - \frac{3}{8(3t^2+1)} + \frac{1}{8(t^2+3)} \right] dt$$

$$= \int_0^1 \frac{dt}{2(t^2+1)} = \int_0^1 \frac{3 dt}{8(3t^2+1)} - \int_0^1 \frac{3 dt}{8(t^2+3)}$$

$$= \left( \frac{1}{2} \tan^{-1} t \right)_0^1 - \left( \frac{3\sqrt{3}}{8} \tan^{-1} \sqrt{3} t \right)_0^1 - \left( \frac{3}{8\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} \right)_0^1$$

$$= \frac{\pi}{8} - \frac{\sqrt{3}\pi}{8 \cdot 3} - \frac{\sqrt{3}\pi}{8 \cdot 6} = \frac{\pi}{8} - \frac{\sqrt{3}\pi}{16} = \frac{\pi}{8} \left( 1 - \frac{\sqrt{3}}{2} \right)$$

## Question 164

The value of  $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left( \left( \frac{x+1}{x-1} \right)^2 + \left( \frac{x-1}{x+1} \right)^2 - 2 \right)^{1/2} dx$  is

[26 Aug 2021 Shift 1]

Options:

A.  $\log_e 4$

B.  $\log_e 16$

C.  $2\log_e 16$

D.  $4\log_e(3 + 2\sqrt{2})$

Answer: B

Solution:

Solution:

$$\text{Let } I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left( \left( \frac{x+1}{x-1} \right)^2 + \left( \frac{x-1}{x+1} \right)^2 - 2 \right)^{1/2} dx$$

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[ \left( \frac{x+1}{x-1} - \frac{x-1}{x+1} \right)^2 \right]^{1/2} dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left| \frac{x+1}{x-1} - \frac{x-1}{x+1} \right| dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left| \frac{(x+1)^2 - (x-1)^2}{(x-1)(x+1)} \right| dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left| \frac{4x}{(x-1)(x+1)} \right| dx$$

$$= 2.4 \int_0^{1/\sqrt{2}} \left| \frac{x}{(x-1)(x+1)} \right| dx = 4 \int_0^{1/\sqrt{2}} \frac{-2x}{x^2-1} dx$$

$$= -4[\log(x^2-1)]_0^{1/\sqrt{2}}$$

$$= -4 \left[ \log\left(\frac{1}{2}-1\right) - \log|-1| \right]$$

$$= -4 \log\left(\frac{1}{2}\right) = 4 \ln 2 = \ln 16$$

---

Question 165

The value of  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{n^2}{n^2 + 4r^2}$  is

[26 Aug 2021 Shift 1]

Options:

A.  $\frac{1}{2} \tan^{-1}(2)$

B.  $\frac{1}{2} \tan^{-1}(4)$

C.  $\tan^{-1}(4)$

D.  $\frac{1}{4} \tan^{-1}(4)$

Answer: B

Solution:

Solution:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{n^2}{n^2 + 4r^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{1}{1 + 4\left(\frac{r}{n}\right)^2} = \int_0^2 \frac{1}{1 + 4x^2} dx \\ &= \frac{1}{2} [\tan^{-1} 2x]_0^2 = \frac{1}{2} \tan^{-1} 4 \end{aligned}$$

---

## Question 166

If the value of the integral  $\int_0^5 \frac{x + [x]}{e^{x - [x]}} dx = \alpha e^{-} + \beta$ , where  $\alpha, \beta \in \mathbb{R}$ ,  $5\alpha + 6\beta = 0$  and

$[x]$  denotes the greatest integer less than or equal to  $x$ , then the value of

$(\alpha + \beta)^2$  is equal to:

[26 Aug 2021 Shift 2]

Options:

A. 100

B. 25

C. 16

D. 36



**Answer: B**

**Solution:**

**Solution:**

$$I = \int_0^5 \frac{x + [x]}{e^{x-[x]}} dx = \alpha e^{-1} + \beta,$$
$$I = \int_0^1 \frac{x}{e^x} dx + \int_1^2 \frac{x+1}{e^{x-1}} dx + \int_2^3 \frac{x+2}{e^{x-2}} dx + \int_3^4 \frac{x+3}{e^{x-3}} dx + \int_4^5 \frac{x+4}{e^{x-4}} dx$$

$$\text{Let } I = I_1 + I_2 + I_3 + I_4 + I_5$$

$$\text{Here, } I_2 = \int_1^2 \frac{x+1}{e^{x-1}} dx \text{ Put } x = t + 1$$

$$\Rightarrow dx = dt$$
$$= \int_0^1 \frac{t+2}{e^t} dt = \int_0^1 \frac{t}{e^t} dt + \int_0^1 \frac{2}{e^t} dt$$

$$I_2 = I_1 + 2 \int_0^1 e^{-t} dt = I_1 + 2(1 - e^{-1})$$

Similarly,

$$I_3 = I_1 + 4(1 - e^{-1})$$

$$I_4 = I_1 + 6(1 - e^{-1})$$

$$I_5 = I_1 + 8(1 - e^{-1})$$

$$I = I_1 + I_2 + I_3 + I_4 + I_5 = 5I_1 + (2 + 4 + 6 + 8)(1 - e^{-1})$$
$$= 5I_1 + 20(1 - e^{-1})$$

$$I_1 = \int_0^1 x e^{-x} dx = -[e^{-x}(x+1)]_0^1 = 1 - 2e^{-1}$$

$$\therefore 5I_1 + 20(1 - e^{-1}) = 5(1 - 2e^{-1}) + 20(1 - e^{-1}) = 25 - 30e^{-1}$$

$$\therefore \alpha = -30, \beta = 25$$

Also it satisfy  $5\beta + 6\alpha = 0$

$$\text{Now, } (\alpha + \beta)^2 = (-30 + 25)^2 = (-5)^2 = 25$$

## Question 167

Let  $J_{n,m} = \int_0^1 \frac{x^n}{x^m - 1} dx$ ,  $\forall n > m$  and  $n, m \in \mathbb{N}$ .

Consider a matrix  $A = [a_{ij}]_{3 \times 3}$  where

$$a_{ij} = \begin{cases} J_{6+i}, & 3 - J_i + 3, & 3 \quad i \leq j \\ 0 & & i > j \end{cases}$$

then  $|\text{adj } A^{-1}|$  is

[1 Sep 2021 Shift 2]

Options:

A.  $(15)^2 \times 2^{42}$

B.  $(15)^2 \times 2^{34}$

C.  $(105)^2 \times 2^{38}$

D.  $(105)^2 \times 2^{36}$

**Answer: C**

**Solution:**

**Solution:**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\Rightarrow |A| = a_{11}a_{22}a_{33}$$

$$\Rightarrow |A| = (J_{7,3} - J_{4,3}) (J_{8,3} - J_{5,3}) (J_{9,3} - J_{6,3})$$

$$= \int_0^1 \frac{1}{2} \frac{x^7 - x^4}{x^3 - 1} dx \cdot \int_0^1 \frac{1}{2} \frac{x^8 - x^5}{x^3 - 1} dx \cdot \int_0^1 \frac{1}{2} \frac{x^9 - x^6}{x^3 - 1} dx$$

$$= \int_0^1 \frac{1}{2} x^4 dx \int_0^1 \frac{1}{2} x^5 dx \cdot \int_0^1 \frac{1}{2} x^6 dx$$

$$= \frac{x^5}{5} \Big|_0^1 \cdot \frac{x^6}{6} \Big|_0^1 \cdot \frac{x^7}{7} \Big|_0^1 = \frac{1}{(210)2^{18}}$$

$$\text{Now, } |\text{adj } A^{-1}| = \frac{1}{|A|^2} = ((210) \cdot 2^{18})^2 = 105^2 \cdot 2^{38}$$

---

## Question 168

The function  $f(x)$ , that satisfies the condition

$$f(x) = x + \int_0^{\pi/2} \sin x \cdot \cos y f(y) dy, \text{ is}$$

[1 Sep 2021 Shift 2]

**Options:**

A.  $x + \frac{2}{3}(\pi - 2) \sin x$

B.  $x + (\pi + 2) \sin x$

C.  $x + \frac{\pi}{2} \sin x$

D.  $x + (\pi - 2) \sin x$

**Answer: D**

**Solution:**

**Solution:**

$$f(x) = x + \int_0^{\frac{\pi}{2}} \sin x \cdot \cos y f(y) dy$$

$$\text{Let } K = \int_0^{\frac{\pi}{2}} \cos y f(y) dy \dots(i)$$

$$\text{Then, } f(x) = x + K \sin x \dots (ii)$$

From Eqs. (i) and (ii),

$$\begin{aligned} f(x) &= x + \int_0^{\frac{\pi}{2}} \sin x \cos y (y + k \sin y) dy \\ &= x + \sin x \int_0^{\frac{\pi}{2}} y \cos y dy + \frac{k}{2} \sin x \int_0^{\frac{\pi}{2}} \sin 2y dy \end{aligned}$$

$$f(x) = x + \sin x \cdot \frac{\pi \cdot 2}{2} + \frac{k \sin x}{2} \dots(iii)$$

$$k = \frac{\pi - 2}{2} + \frac{k}{2}$$

$$\Rightarrow k = \pi - 2$$

$$\therefore f(x) = x + (\pi - 2) \sin x$$

---

## Question 169

The integral  $\int \frac{dx}{(x+4)^{8/7}(x-3)^{6/7}}$  is equal to: (where C is a constant of integration)

[Jan. 9, 2020 (I)]

**Options:**

A.  $\left(\frac{x-3}{x+4}\right)^{1/7} + C$

B.  $-\left(\frac{x-3}{x+4}\right)^{1/7} + C$

C.  $\frac{1}{2}\left(\frac{x-3}{x+4}\right)^{3/7} + C$

D.  $-\frac{1}{13}\left(\frac{x-3}{x+4}\right)^{-13/7} + C$

**Answer: A**

## Solution:

### Solution:

$$I = \int \frac{dx}{(x+4)^{8/7}(x-3)^{6/7}}$$
$$= \int \left(\frac{x-3}{x+4}\right)^{-6} \frac{1}{(x+4)^2} dx$$

$$\text{Let } \frac{x-3}{x+4} = t^7$$

Differentiate on both sides, we get

$$\frac{7}{(x+4)^2} dx = 7t^6 dt$$

$$\text{Hence, } I = \int t^{-6} t^6 dt = t + C = \left(\frac{x-3}{x+4}\right)^{\frac{1}{7}} + C$$

---

## Question 170

If  $\int \frac{d\theta}{\cos^2\theta(\tan 2\theta + \sec 2\theta)} = \lambda \tan \theta + 2 \log_e |f(\theta)| + C$  where  $C$  is a constant of integration, then the ordered pair  $(\lambda, f(\theta))$  is equal to:  
[Jan. 9, 2020 (II)]

### Options:

- A.  $(1, 1 - \tan \theta)$
- B.  $(-1, 1 - \tan \theta)$
- C.  $(-1, 1 + \tan \theta)$
- D.  $(1, 1 + \tan \theta)$

**Answer: C**

## Solution:

### Solution:

$$I = \int \frac{d\theta}{\cos^2\theta(\tan 2\theta + \sec 2\theta)}$$
$$= \int \frac{\sec^2\theta}{\frac{1 + \tan^2\theta}{1 - \tan^2\theta} + \frac{2 \tan \theta}{1 - \tan^2\theta}} d\theta$$
$$= \int \frac{\sec^2\theta(1 - \tan^2\theta)}{(1 + \tan \theta)^2} d\theta$$
$$= \int \frac{\sec^2\theta(1 - \tan \theta)}{1 + \tan \theta} d\theta$$

Let  $\tan \theta = t \Rightarrow \sec^2 \theta d\theta = dt$ , then

$$I = \int \left( \frac{1-t}{1+t} \right) dt = \int \left( -1 + \frac{2}{1+t} \right) dt$$

$$= -t + 2 \log(1+t) + C$$

$$= -\tan \theta + 2 \log(1 + \tan \theta) + C$$

Hence, by comparison  $\lambda = -1$  and  $f(x) = 1 + \tan \theta$

---

## Question 171

If  $\int \frac{\cos x dx}{\sin^3 x (1 + \sin^6 x)^{2/3}} = f(x)(1 + \sin^6 x)^{1/\lambda} + c$  where  $c$  is a constant of integration,

then  $\lambda f\left(\frac{\pi}{3}\right)$  is equal to:

[Jan. 8, 2020 (II)]

Options:

A.  $-\frac{9}{8}$

B. 2

C.  $\frac{9}{8}$

D. -2

Answer: D

Solution:

Solution:

$$\text{Let } I = \int \frac{\cos x dx}{\sin^3 x (1 + \sin^6 x)^{2/3}}$$

$$= f(x)(1 + \sin^6 x)^{1/\lambda} + c \dots (i)$$

If  $\sin x = t$

then,  $\cos x dx = dt$

$$I = \int \frac{dt}{t^3(1+t^6)^{2/3}} = \int \frac{dt}{t^7 \left(1 + \frac{1}{t^6}\right)^{2/3}}$$

$$\text{Put } 1 + \frac{1}{t^6} = r^3 \Rightarrow \frac{dt}{t^7} = \frac{-1}{2} r^2 dr - \frac{1}{2} \int \frac{r^2 dr}{r^2} = -\frac{1}{2} r + c$$

$$= -\frac{1}{2} \left( \frac{\sin^6 x + 1}{\sin^6 x} \right)^{1/3} + c = -\frac{1}{2 \sin^2 x} (1 + \sin^6 x)^{1/3} + c$$



$$f(x) = -\frac{1}{2}\operatorname{cosec}^2 x \text{ and } \lambda = 3[\text{from eqn. (i)}]$$

$$\therefore \lambda f\left(\frac{\pi}{3}\right) = -2$$

---

## Question 172

If for all real triplets (a, b, c),  $f(x) = a + bx + cx^2$ ; then  $\int_0^1 f(x) dx$  is equal to:  
[Jan. 9, 2020 (I)]

Options:

A.  $2 \left\{ 3f(1) + 2f\left(\frac{1}{2}\right) \right\}$

B.  $\frac{1}{2} \left\{ f(1) + 3f\left(\frac{1}{2}\right) \right\}$

C.  $\frac{1}{3} \left\{ f(0) + f\left(\frac{1}{2}\right) \right\}$

D.  $\frac{1}{6} \left\{ f(0) + f(1) + 4f\left(\frac{1}{2}\right) \right\}$

Answer: D

Solution:

Solution:

$$\int_0^1 (a + bx + cx^2) dx = ax + \frac{bx^2}{2} + \frac{cx^3}{3} \Big|_0^1 = a + \frac{b}{2} + \frac{c}{3}$$

Now,  $f(1) = a + b + c$ ,  $f(0) = a$  and  $f\left(\frac{1}{2}\right) = a + \frac{b}{2} + \frac{c}{4}$

$$\begin{aligned} \text{Now, } & \frac{1}{6} \left( f(1) + f(0) + 4f\left(\frac{1}{2}\right) \right) \\ &= \frac{1}{6} \left( a + b + c + a + 4 \left( a + \frac{b}{2} + \frac{c}{4} \right) \right) \\ &= \frac{1}{6} (6a + 3b + 2c) = a + \frac{b}{2} + \frac{c}{3} \end{aligned}$$

Hence,  $\int_0^1 f(x) dx = \frac{1}{6} \left\{ f(0) + f(1) + 4f\left(\frac{1}{2}\right) \right\}$

---

## Question 173

The value of  $\int_0^{2\pi} \frac{x \sin^8 x}{\sin^8 x + \cos^8 x} dx$  is equal to:

[Jan. 9, 2020 (I)]

Options:

A.  $2\pi$

B.  $2\pi^2$

C.  $\pi^2$

D.  $4\pi$

Answer: C

Solution:

$$\begin{aligned} & \int_0^{2\pi} \frac{x \sin^8 x}{\sin^8 x + \cos^8 x} dx \\ &= \int_0^{\pi} \left[ \frac{x \sin^8 x}{\sin^8 x + \cos^8 x} + \frac{(2\pi - x) \sin^8 x}{\sin^8 x + \cos^8 x} \right] dx \left[ \because \int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a - x) dx \right] \\ &= \int_0^{\pi} \frac{2\pi \sin^8 x}{\sin^8 x + \cos^8 x} dx \\ &= 2\pi \int_0^{\pi/2} \left[ \frac{\sin^8 x}{\sin^8 x + \cos^8 x} + \frac{\cos^8 x}{\sin^8 x + \cos^8 x} \right] dx \\ &= 2\pi \int_0^{\pi/2} 1 dx = 2\pi \times \frac{\pi}{2} = \pi^2 \end{aligned}$$

---

## Question 174

If  $I = \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$ , then:

[Jan. 8, 2020 (II)]

Options:

A.  $\frac{1}{8} < I^2 < \frac{1}{4}$

B.  $\frac{1}{9} < I^2 < \frac{1}{8}$

C.  $\frac{1}{16} < I^2 < \frac{1}{9}$

$$D. \frac{1}{6} < I^2 < \frac{1}{2}$$

**Answer: B**

**Solution:**

**Solution:**

$$f(x) = \frac{1}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$$

$$f'(x) = \frac{-1}{2} \left( \frac{(6x^2 - 18x + 12)}{(2x^3 - 9x^2 + 12x + 4)^{3/2}} \right)$$

$$= \frac{-6(x-1)(x-2)}{2(2x^3 - 9x^2 + 12x + 4)^{3/2}}$$

$$f(1) = \frac{1}{3} \text{ and } f(2) = \frac{1}{\sqrt{8}}$$

It is increasing function

$$\frac{1}{3} < I < \frac{1}{\sqrt{8}}$$

$$\frac{1}{9} < I^2 < \frac{1}{8}$$

---

## Question 175

If  $f(a + b + 1 - x) = f(x)$ , for all  $x$ , where  $a$  and  $b$  are fixed positive real numbers, then  $\frac{1}{a+b} \int_a^b x(f(x) + f(x+1)) dx$  is equal to:

**[Jan. 7, 2020 (I)]**

**Options:**

A.  $\int_{a+1}^{b+1} f(x) dx$

B.  $\int_{a-1}^{b-1} f(x) dx$

C.  $\int_{a-1}^{b-1} f(x+1) dx$

D.  $\int_{a+1}^{b+1} f(x+1) dx$

**Answer: C**

**Solution:**

$$I = \frac{1}{(a+b)} \int_a^b x[f(x) + f(x+1)] dx \dots\dots(i)$$

$$x \rightarrow a+b-x$$

$$I = \frac{1}{(a+b)} \int_a^b (a+b-x)[f(a+b-x) + f(a+b+1-x)] dx$$

$$I = \frac{1}{(a+b)} \int_a^b (a+b-x)[f(x+1) + f(x)] dx \dots\dots(ii)$$

[∵ put  $x \rightarrow x+1$  in  $f(a+b+1-x) = f(x)$ ]

Add (i) and (ii)

$$2I = \int_a^b [f(x+1) + f(x)] dx$$

$$2I = \int_a^b f(x+1) dx + \int_a^b f(x) dx$$

$$= \int_a^b f(a+b+1-x) dx + \int_a^b f(x) dx$$

$$2I = 2 \int_a^b f(x) dx$$

$$\therefore \int_{a-1}^{b-1} f(x+1) dx \quad [\because \text{Put } x \rightarrow x+1]$$

## Question 176

The value of  $\alpha$  for which  $4\alpha \int_{-1}^2 e^{-\alpha|x|} dx = 5$ , is :

[Jan. 7, 2020 (II)]

Options:

A.  $\log_e 2$

B.  $\log_e \left(\frac{3}{2}\right)$

C.  $\log_e \sqrt{2}$

D.  $\log_e \left(\frac{4}{3}\right)$

**Answer: A**

**Solution:**

$$4\alpha \left\{ \int_{-1}^0 e^{\alpha x} dx + \int_0^2 e^{-\alpha x} dx \right\} = 5$$

$$\Rightarrow 4\alpha \left\{ \frac{e^{\alpha x}}{\alpha} \Big|_{-1}^0 + \frac{e^{-\alpha x}}{-\alpha} \Big|_0^2 \right\} = 5$$

$$\Rightarrow 4\alpha \left\{ \left( \frac{1 - e^{-\alpha}}{\alpha} \right) - \left( \frac{e^{-2\alpha} - 1}{\alpha} \right) \right\} = 5$$

$$\Rightarrow 4(2 - e^{-\alpha} - e^{-2\alpha}) = 5$$

Put  $e^{-\alpha} = t$

$$\Rightarrow 4t^2 + 4t - 3 = 0 \Rightarrow (2t + 3)(2t - 1) = 0$$

$$\Rightarrow e^{-\alpha} = \frac{1}{2} \Rightarrow \alpha = \log_e 2$$


---

## Question 177

If  $\theta_1$  and  $\theta_2$  be respectively the smallest and the largest values of theta in  $(0, 2\pi) - \{\pi\}$  which satisfy the equation,  $2\cot^2\theta - \frac{5}{\sin\theta} + 4 = 0$ , then  $\int_{\theta_1}^{\theta_2} \cos^2 3\theta d\theta$  is equal to:

[Jan. 7, 2020 (II)]

Options:

- A.  $\frac{\pi}{3}$
- B.  $\frac{2\pi}{3}$
- C.  $\frac{\pi}{3} + \frac{1}{6}$
- D.  $\frac{\pi}{9}$

**Answer: A**

**Solution:**

**Solution:**

$$2\cot^2\theta - \frac{5}{\sin\theta} + 4 = 0$$

$$\frac{2\cos^2\theta}{\sin^2\theta} - \frac{5}{\sin\theta} + 4 = 0$$

$$\Rightarrow 2\cos^2\theta - 5\sin\theta + 4\sin^2\theta = 0, \sin\theta \neq 0$$

$$\Rightarrow 2\sin^2\theta - 5\sin\theta + 2 = 0$$

$$\Rightarrow (2\sin\theta - 1)(\sin\theta - 2) = 0$$



$$\begin{aligned} \therefore \sin \theta = \frac{1}{2} &\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6} \\ \therefore \int_{\pi/6}^{5\pi/6} \cos^2 3\theta d\theta &= \int_{\pi/6}^{5\pi/6} \frac{1 + \cos 6\theta}{2} d\theta \\ &= \frac{1}{2} \left[ \theta + \frac{\sin 6\theta}{6} \right]_{\pi/6}^{5\pi/6} = \frac{1}{2} \left[ \frac{5\pi}{6} - \frac{\pi}{6} + \frac{1}{6}(0 - 0) \right] \\ &= \frac{1}{2} \cdot \frac{4\pi}{6} = \frac{\pi}{3} \end{aligned}$$


---

## Question 178

Let a function  $f : [0, 5] \rightarrow \mathbb{R}$  be continuous,  $f(1) = 3$  and  $F$  be defined as:

$$F(x) = \int_1^x t^2 g(t) dt, \text{ where } g(t) = \int_1^x f(u) du$$

Then for the function  $F$ , the point  $x = 1$  is:

[Jan. 9, 2020 (II)]

Options:

- A. a point of local minima.
- B. not a critical point.
- C. a point of local maxima.
- D. a point of inflection.

**Answer: A**

**Solution:**

**Solution:**

$$F(x) = \int_1^x t^2 g(t) dt$$

Differentiate by using Leibnitz's rule, we get

$$F'(x) = x^2 g(x) = x^2 \int_1^x f(u) du \dots\dots(i)$$

At  $x = 1$

$$F'(1) = 1 \int_1^1 f(u) du = 0$$

Now, differentiate eqn (i)

$$F''(x) = x^2 f(x) - 2x \int_1^x f(u) du$$

At  $x = 1$ ,

$$F''(1) = 1 \cdot f(1) - 2 \times 1 \cdot \int_1^1 f(u) du$$

$$= f(1) - 2 \times 0 = f(1)$$

$$F''(1) = 3$$

Then, for  $F'(1) = 0$ ,  $F''(1) = 3 > 0$   
Hence,  $x = 1$  is a point of local minima.

---

## Question 179

$$\lim_{x \rightarrow 1} \left( \frac{\int_0^{(x-1)^2} t \cos(t^2) dt}{(x-1) \sin(x-1)} \right)$$

[Sep. 06, 2020 (I)]

Options:

- A. is equal to  $\frac{1}{2}$
- B. is equal to 1
- C. is equal to  $-\frac{1}{2}$
- D. (Bonus)

Answer: D

Solution:

Solution:

$$\lim_{x \rightarrow 1} \frac{\frac{1}{2} \sin(x-1)^4}{(x-1) \sin(x-1)}$$

Let  $x - 1 = h$  when  $x \rightarrow 1$  then  $h \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{\sin h^4}{h^4} \times \frac{h}{\sin h} \times h^2 = 1 \times 1 \times 0 = 0$$

(No any option is correct)

---

## Question 180

If  $\int (e^{2x} + 2e^x - e^{-x} - 1)e^{(e^x + e^{-x})} dx = g(x)e^{(e^x + e^{-x})} + c$ , where  $c$  is a constant of integration, then  $g(0)$  is equal to:

[Sep. 05, 2020 (I)]

Options:

A. e

B. e<sup>2</sup>

C. 1

D. 2

**Answer: D**

**Solution:**

**Solution:**

$$\int (e^{2x} + 2e^x - e^{-x} - 1) \cdot e^{(e^x + e^{-x})} dx$$

$$I = \int (e^{2x} + e^x - 1) \cdot e^{(e^x + e^{-x})} dx + \int (e^x - e^{-x}) e^{(e^x + e^{-x})} dx$$

$$= \int e^x (e^x + 1 - e^{-x}) \cdot e^{(e^x + e^{-x})} dx + e^{(e^x + e^{-x})}$$

$$= \int (e^x - e^{-x} + 1) e^{(e^x + e^{-x} + x)} dx + e^{(e^x + e^{-x})}$$

$$\text{Let } e^x + e^{-x} + x = t \Rightarrow (e^x + e^{-x} + 1) dx = dt$$

$$= \int e^t dt + e^{(e^x + e^{-x})} = e^t + e^{(e^x + e^{-x})} + C$$

$$= e^{(e^x + e^{-x} + x)} + e^{(e^x + e^{-x})} + C$$

$$= (e^x + 1) \cdot e^{(e^x + e^{-x})} + C$$

$$\text{So, } g(x) = 1 + e^x \text{ and } g(0) = 2$$

---

## Question 181

If  $\int \frac{\cos \theta}{5 + 7 \sin \theta - 2 \cos^2 \theta} d\theta = A \log_e |B(\theta)| + C$ , where C is a constant of integration, then  $\frac{B(\theta)}{A}$  can be:

[Sep. 05, 2020 (II)]

**Options:**

A.  $\frac{2 \sin \theta + 1}{\sin \theta + 3}$

B.  $\frac{2 \sin \theta + 1}{5(\sin \theta + 3)}$

C.  $\frac{5(\sin \theta + 3)}{2 \sin \theta + 1}$

D.  $\frac{5(2 \sin \theta + 1)}{\sin \theta + 3}$

**Answer: D**



## Solution:

### Solution:

Let  $\sin \theta = t \Rightarrow \cos \theta d\theta = dt$

$$\int \frac{\cos \theta}{5 + 7 \sin \theta - 2 \cos^2 \theta} d\theta = \frac{dt}{5 + 7t - 2 + 2t^2}$$

$$\Rightarrow \frac{1}{2} \int \frac{dt}{\left(t + \frac{7}{4}\right)^2 - \left(\frac{5}{4}\right)^2} = \frac{1}{5} \ln \left| \frac{t + \frac{1}{2}}{t + 3} \right| + C$$

$$= \frac{1}{5} \ln \left| \frac{2t + 1}{t + 3} \right| + C = \frac{1}{5} \ln \left| \frac{2 \sin \theta + 1}{\sin \theta + 3} \right| + C$$

$$\therefore B(\theta) = \frac{2 \sin \theta + 1}{2(\sin \theta + 3)} \text{ and } A = \frac{1}{5}$$

$$\Rightarrow \frac{B(\theta)}{A} = \frac{5(2 \sin \theta + 1)}{(\sin \theta + 3)}$$

---

## Question 182

The integral  $\int \left( \frac{x}{x \sin x + \cos x} \right)^2 dx$  is equal to (where  $C$  is a constant of integration):  
[Sep. 04, 2020 (I)]

### Options:

A.  $\tan x - \frac{x \sec x}{x \sin x + \cos x} + C$

B.  $\sec x + \frac{x \tan x}{x \sin x + \cos x} + C$

C.  $\sec x - \frac{x \tan x}{x \sin x + \cos x} + C$

D.  $\tan x + \frac{x \sec x}{x \sin x + \cos x} + C$

**Answer: A**

### Solution:

#### Solution:

$$\int \frac{x^2}{(x \sin x + \cos x)^2} dx$$

$$\therefore \frac{d}{dx}(x \sin x + \cos x) = x \cos x$$

$$= \int \frac{x \cos x}{(x \sin x + \cos x)^2} \left( \frac{x}{\cos x} \right) dx$$

$$\begin{aligned}
&= \frac{x}{\cos x} \left[ \frac{-1}{x \sin x + \cos x} \right] - \int \frac{x \sin x + \cos x}{\cos^2 x} \left[ \frac{-1}{x \sin x + \cos x} \right] dx \\
&= \frac{x}{\cos x} \left[ \frac{-1}{x \sin x + \cos x} \right] + \int \sec^2 x dx \\
&= \frac{-x \sec x}{x \sin x + \cos x} + \tan x + C
\end{aligned}$$


---

## Question 183

Let  $f(x) = \int \frac{\sqrt{x}}{(1+x)^2} dx$  ( $x \geq 0$ ). Then  $f(3) - f(1)$  is equal to :

[Sep. 04, 2020 (I)]

Options:

A.  $-\frac{\pi}{12} + \frac{1}{2} + \frac{\sqrt{3}}{4}$

B.  $\frac{\pi}{6} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

C.  $-\frac{\pi}{6} + \frac{1}{2} + \frac{\sqrt{3}}{4}$

D.  $\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

Answer: D

Solution:

Solution:

$$\int \frac{\sqrt{x}}{(1+x)^2} dx \quad (x > 0)$$

$$\text{Put } x = \tan^2 \theta \Rightarrow 2x dx = 2 \tan \theta \sec^2 \theta d\theta$$

$$I = \int \frac{2 \tan^2 \theta \cdot \sec^2 \theta}{\sec^4 \theta} d\theta = \int 2 \sin^2 \theta d\theta$$

$$= \theta - \frac{\sin 2\theta}{2} + C$$

$$\Rightarrow f(x) = \theta - \frac{1}{2} \times \frac{2 \tan \theta}{1 + \tan^2 \theta} + C$$

$$f(x) = \theta - \frac{\tan \theta}{1 + \tan^2 \theta} + C = \tan^{-1} \sqrt{x} - \frac{\sqrt{x}}{1+x} + C$$

$$\text{Now } f(3) - f(1) = \tan^{-1}(\sqrt{3}) - \frac{\sqrt{3}}{1+3} - \tan^{-1}(1) + \frac{1}{2}$$

$$= \frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}$$


---

## Question 184

If  $\int \sin^{-1} \left( \sqrt{\frac{x}{1+x}} \right) dx = A(x)\tan^{-1}(\sqrt{x}) + B(x) + C$ , where  $C$  is a constant of integration, then the ordered pair  $(A(x), B(x))$  can be :  
[Sep. 03, 2020 (II)]

Options:

A.  $(x + 1, -\sqrt{x})$

B.  $(x + 1, \sqrt{x})$

C.  $(x - 1, -\sqrt{x})$

D.  $(x - 1, \sqrt{x})$

Answer: A

Solution:

Solution:

$$\begin{aligned} I &= \int \sin^{-1} \left( \frac{\sqrt{x}}{\sqrt{1+x}} \right) dx = \int \tan^{-1} \sqrt{x} \cdot 1 dx \\ &= x \tan^{-1} \sqrt{x} - \int \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} \cdot x dx + C \\ &= x \tan^{-1} \sqrt{x} - \frac{1}{2} \int \frac{t \cdot 2tdt}{1+t^2} + C \quad (\text{Put } x = t^2 \Rightarrow dx = 2tdt) \\ &= x \tan^{-1} \sqrt{x} - \int \frac{t^2}{1+t^2} dt + C \\ &= x \tan^{-1} \sqrt{x} - t + \tan^{-1} t + C \\ &= x \tan^{-1} \sqrt{x} - \sqrt{x} + \tan^{-1} \sqrt{x} + C \\ &= (x+1) \tan^{-1} \sqrt{x} - \sqrt{x} + C \\ &\Rightarrow A(x) = x+1 \Rightarrow B(x) = -\sqrt{x} \end{aligned}$$

---

## Question 185

The integral  $\int_1^2 e^x \cdot x^x (2 + \log_e x) dx$  equals:  
[Sep. 06, 2020 (II)]

Options:



A.  $e(4e + 1)$

B.  $4e^2 - 1$

C.  $e(4e - 1)$

D.  $e(2e - 1)$

**Answer: C**

**Solution:**

**Solution:**

$$I = \int_1^2 e^x x^x (2 + \log_e x) dx$$

$$I = \int_1^2 e^x x^x [1 + (1 + \log_e x)] dx$$

$$= \int_1^2 e^x [x^x + x^x (1 + \log_e x)] dx$$

$$\because \int e^x (f(x) + f'(x)) dx = e^x f(x) + c$$

$$\therefore I = [e^x x^x]_1^2$$

$$= e^2 \times 4 - e \times 1 = 4e^2 - e = e(4e - 1)$$

---

## Question 186

If  $I_1 = \int_0^1 (1 - x^{50})^{100} dx$  and  $I_2 = \int_0^1 (1 - x^{50})^{101} dx$  such that  $I_2 = \alpha I_1$  then  $\alpha$  equals to :

[Sep. 06, 2020 (I)]

**Options:**

A.  $\frac{5049}{5050}$

B.  $\frac{5050}{5049}$

C.  $\frac{5050}{5051}$

D.  $\frac{5051}{5050}$

**Answer: C**

**Solution:**

$$\begin{aligned}
I_2 &= \int_0^1 (1-x^{50})^{101} dx = \int_0^1 (1-x^{50})(1-x^{50})^{100} dx \\
I_2 &= \int_0^1 (1-x^{50})^{100} dx - \int_0^1 x \cdot x^{49} (1-x^{50})^{100} dx \\
I_2 &= I_1 + \left[ \frac{x}{5050} (1-x^{50})^{101} \right]_0^1 - \int_0^1 \frac{(1-x^{50})^{101}}{5050} dx \\
I_2 &= I_1 + 0 - \frac{I_2}{5050} \\
\Rightarrow \frac{5051}{5050} I_2 &= I_1 \Rightarrow I_2 = \frac{5050}{5051} I_1 \\
\Rightarrow \alpha &= \frac{5050}{5051}
\end{aligned}$$


---

## Question 187

The value of  $\int_{-\pi/2}^{\pi/2} \frac{1}{1+e^{\sin x}} dx$  is:

[Sep. 05, 2020 (I)]

Options:

- A.  $\frac{\pi}{4}$
- B.  $\pi$
- C.  $\frac{\pi}{2}$
- D.  $\frac{3\pi}{2}$

**Answer: C**

**Solution:**

**Solution:**

$$\begin{aligned}
I &= \int_{-\pi/2}^{\pi/2} \frac{1}{1+e^{\sin x}} dx \\
&= \int_{-\pi/2}^0 \frac{1}{1+e^{\sin x}} dx + \int_0^{\pi/2} \frac{1}{1+e^{\sin x}} dx \\
&= \int_0^{\pi/2} \left( \frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}} \right) dx \\
&= \int_0^{\pi/2} \frac{1+e^{\sin x}}{1+e^{\sin x}} dx = \frac{\pi}{2}
\end{aligned}$$


---



## Question188

Let  $f(x) = |x - 2|$  and  $g(x) = f(f(x))$ ,  $x \in [0, 4]$  Then  $\int_0^3 (g(x) - f(x)) dx$  is equal to:

[Sep. 04, 2020 (I)]

Options:

A. 1

B. 0

C.  $\frac{1}{2}$

D.  $\frac{3}{2}$

Answer: A

Solution:

Solution:

$$f(x) = |x - 2| = \begin{cases} 2 - x, & x < 2 \\ x - 2, & x \geq 2 \end{cases}$$

$$g(x) = f(f(x)) = \begin{cases} 2 - f(x), & f(x) < 2 \\ f(x) - 2, & f(x) \geq 2 \end{cases}$$

$$= \begin{cases} 2 - (2 - x), & 2 - x < 2, & x < 2 \\ (2 - x) - 2, & 2 - x \geq 2, & x < 2 \\ 2 - (x - 2), & x - 2 < 2, & x \geq 2 \\ (x - 2) - 2, & x - 2 \geq 2, & x \geq 2 \end{cases}$$

$$= \begin{cases} -x, & 0 < x \leq 0 \\ x, & 0 < x < 2 \\ 4 - x, & 2 \leq x < 4 \\ x - 4, & x \geq 4 \end{cases}$$

$$\therefore \int_0^3 [g(x) - f(x)] dx$$

$$= \int_0^2 x dx + \int_2^3 (4 - x) dx - \int_0^3 |x - 2| dx = 1$$

---

## Question189

The integral  $\int_{\pi/6}^{\pi/3} \tan^3 x \cdot \sin^2 3x (2 \sec^2 x \cdot \sin^2 3x + 3 \tan x \cdot \sin 6x) dx$  is equal to :  
 [Sep. 04, 2020 (II)]

Options:

A.  $\frac{7}{18}$

B.  $-\frac{1}{9}$

C.  $-\frac{1}{18}$

D.  $\frac{9}{2}$

Answer: C

Solution:

Solution:

$$\begin{aligned} & \int_{\pi/6}^{\pi/3} \left[ \frac{1}{2} \frac{d(\tan^4 x)}{dx} \cdot \sin^4 3x + \frac{1}{2} \tan^4 x \cdot \frac{d(\sin^4 3x)}{dx} \right] dx \\ &= \frac{1}{2} \int_{\pi/6}^{\pi/3} d(\tan^4 x \cdot \sin^4 3x) dx \\ &= \left[ \frac{\tan^4 x \sin^4 3x}{2} \right]_{\pi/6}^{\pi/3} = \frac{9 \cdot 0}{2} - \frac{\frac{1}{9} \cdot 1}{2} = -\frac{1}{18} \end{aligned}$$

## Question 190

Let  $\{x\}$  and  $[x]$  denote the fractional part of  $x$  and the greatest integer less than or equal to  $x$  respectively of a real number  $x$ . If  $\int_0^n \{x\} dx$ ,  $\int_0^n [x] dx$  and  $10(n^2 - n)$ , ( $n \in \mathbb{N}$ ,  $n > 1$ ) are three consecutive terms of a G.P., then  $n$  is equal to \_\_\_\_\_.  
 [NA Sep. 04, 2020 (II)]

Answer: 21

Solution:

$$\int_0^n \{x\} dx = n \int_0^1 x \cdot dx = \frac{n}{2}$$

$$\Rightarrow \int_0^n [x] dx = \int_0^n (x - \{x\}) dx = \frac{n^2}{2} - \frac{n}{2}$$

According to the questions,

$$\frac{n}{2}, \frac{n^2 - n}{2}, 10(n^2 - n) \text{ are in GP}$$

$$\therefore \left(\frac{n^2 - n}{2}\right)^2 = \frac{n}{2} \times 10(n^2 - n)$$

$$\Rightarrow n^2 = 21n \Rightarrow n = 21$$

---

## Question 191

$\int_{-\pi}^{\pi} |\pi - |x|| dx$  is equal to :

[Sep. 03, 2020 (I)]

Options:

A.  $\sqrt{2}\pi^2$

B.  $2\pi^2$

C.  $\pi^2$

D.  $\frac{\pi^2}{2}$

Answer: C

Solution:

Solution:

$$I = \int_{-\pi}^{\pi} |\pi - |x|| dx \quad [\because |\pi - |x|| \text{ is even}]$$

$$= 2 \int_0^{\pi} |\pi - |x|| dx$$

$$= 2 \int_0^{\pi} (\pi - x) dx$$

$$= 2 \left[ \pi x - \frac{x^2}{2} \right]_0^{\pi} = 2 \left( \pi^2 - \frac{\pi^2}{2} \right) = \pi^2.$$

---

## Question 192



If the value of the integral  $\int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}} dx$  is  $\frac{k}{6}$ , then k is equal to:

[Sep. 03, 2020 (II)]

Options:

A.  $2\sqrt{3} - \pi$

B.  $2\sqrt{3}_\pi$

C.  $3\sqrt{2} + \pi$

D.  $3\sqrt{2} - \pi$

Answer: A

Solution:

Solution:

$$\frac{k}{6} = \int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}} dx$$

$$\text{Let } x = \sin \theta; dx = \cos \theta d\theta$$

$$\text{then } \int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}} dx = \int_0^{\pi/6} \frac{\sin^2 \theta \cos \theta}{\cos^3 \theta} d\theta$$

$$\therefore \frac{k}{6} = \int_0^{\pi/6} \frac{\sin^2 \theta}{\cos^3 \theta} \cdot \cos \theta d\theta$$

$$\Rightarrow \frac{k}{6} = \int_0^{\pi/6} \tan^2 \theta d\theta = \int_0^{\pi/6} (\sec^2 \theta - 1) d\theta$$

$$\Rightarrow \frac{k}{6} = (\tan \theta - \theta)_0^{\pi/6} = \left( \frac{1}{\sqrt{3}} - \frac{\pi}{6} \right) = \frac{2\sqrt{3} - \pi}{6}$$

$$\Rightarrow k = 2\sqrt{3} - \pi$$

---

## Question 193

The integral  $\int_0^2 ||x-1| - x| dx$  is equal to \_\_\_\_\_.

[NA Sep. 02, 2020 (I)]

**Answer: 1.50**

**Solution:**

**Solution:**

$$\begin{aligned} \int_0^2 ||x-1|-x| dx &= \int_0^1 |1-x-x| dx + \int_1^2 |x-1-x| dx \\ &= \int_0^1 (1-2x) dx + \int_{1/2}^1 (2x-1) dx + \int_1^2 dx \\ &= [x-x^2]_0^1 + [x^2-x]_{1/2}^1 + [x]_1^2 \\ &= \frac{1}{2} - \frac{1}{4} + (1-1) - \left(\frac{1}{4} - \frac{1}{2}\right) + 2-1 = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2} \end{aligned}$$

---

## Question 194

Let  $[t]$  denote the greatest integer less than or equal to  $t$ .

Then the value of  $\int_1^2 |2x - [3x]| dx$  is \_\_\_\_\_.

[NA Sep. 02, 2020 (II)]

**Answer: 1**

**Solution:**

**Solution:**

$$\begin{aligned} \int_1^2 |2x - [3x]| dx &= \int_1^2 |3x - [3x] - x| dx \\ &= \int_1^2 |\{3x\} - x| dx = \int_1^2 (x - \{3x\}) dx \\ &= \int_1^2 x dx - \int_1^2 \{3x\} dx \\ &= \left[\frac{x^2}{2}\right]_1^2 - 3 \int_0^1 3x dx \\ &= \frac{(4-1)}{2} - 9 \left[\frac{x^2}{2}\right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1 \end{aligned}$$

---

## Question195

For  $x^2 \neq n\pi + 1$ ,  $n \in \mathbf{N}$  (the set of natural numbers), the integral

$$\int x \sqrt{\frac{2 \sin(x^2 - 1) - \sin 2(x^2 - 1)}{2 \sin(x^2 - 1) + \sin 2(x^2 - 1)}} dx \text{ is equal to:}$$

[Jan. 09, 2019(I)]

Options:

A.  $\log_e \left| \frac{1}{2} \sec^2(x^2 - 1) \right| + c$

B.  $\frac{1}{2} \log_e \left| \sec^2(x^2 - 1) \right| + c$

C.  $\frac{1}{2} \log_e \left| \sec^2 \left( \frac{x^2 - 1}{2} \right) \right| + c$

D.  $\log_e \left| \sec^2 \left( \frac{x^2 - 1}{2} \right) \right| + c$

Answer: 0

Solution:

Solution:

Consider the given integral

$$I = \int x \sqrt{\frac{2 \sin(x^2 - 1) - 2 \sin(x^2 - 1) \cos(x^2 - 1)}{2 \sin(x^2 - 1) + 2 \sin(x^2 - 1) \cos(x^2 - 1)}} dx$$

$$(\because \sin 2\theta = 2 \sin \theta \cos \theta)$$

$$\Rightarrow I = \int x \sqrt{\frac{1 - \cos(x^2 - 1)}{1 + \cos(x^2 - 1)}} dx$$

$$\text{Now let } \frac{x^2 - 1}{2} = t \Rightarrow \frac{2x}{2} dx = dt$$

$$\therefore I = \int |\tan(t)| dt = \ln |\sec t| + C$$

$$\text{or } I = \ln \left| \sec \left( \frac{x^2 - 1}{2} \right) \right| + c = \frac{1}{2} \ln \left| \sec^2 \left( \frac{x^2 - 1}{2} \right) \right| + c$$

---

## Question196

If  $f(x) = \int \frac{5x^8 + 7x^6}{(x^2 + 1 + 2x^7)^2} dx$ , ( $x \geq 0$ ) and  $f(0) = 0$ , then the value of  $f(1)$  is:

[Jan. 09, 2019 (II)]

Options:

A.  $-\frac{1}{2}$

B.  $-\frac{1}{4}$

C.  $\frac{1}{2}$

D.  $\frac{1}{4}$

Answer: D

Solution:

Solution:

$$f(x) = \int \frac{5x^8 + 7x^6}{(x^2 + 1 + 2x^7)^2} dx, x \geq 0$$

$$= \int \frac{5x^8 + 7x^6}{x^{14}(x^{-5} + x^{-7} + 2)^2} dx$$

$$= \int \frac{5x^{-6} + 7x^{-8}}{(2 + x^{-7} + x^{-5})^2} dx$$

$$\text{Let } 2 + x^{-7} + x^{-5} = t$$

$$\Rightarrow (-7x^{-8} - 5x^{-6})dx = dt$$

$$\Rightarrow f(x) = \int \frac{-dt}{t^2} = \int -t^{-2} dt = t^{-1} + c$$

$$\Rightarrow f(x) = \frac{1}{2 + x^{-7} + x^{-5}} + c, f(0) = 0 \Rightarrow c = 0$$

$$\therefore f(1) = \frac{1}{4}$$

---

## Question 197

The integral  $\int \cos(\log_e x) dx$  is equal to :

(where C is a constant of integration)

[Jan. 12, 2019 (I)]

Options:

A.  $\frac{x}{2}[\sin(\log_e x) - \cos(\log_e x)] + C$

B.  $x[\cos(\log_e x) + \sin(\log_e x)] + C$

C.  $\frac{x}{2}[\cos(\log_e x) + \sin(\log_e x)] + C$

D.  $x[\cos(\log_e x) - \sin(\log_e x)] + C$

**Answer: C**

**Solution:**

**Solution:**

Let the integral,  $I = \int \cos(\ln x) dx$

$$\Rightarrow I = \cos(\ln x) x - \int \frac{-\sin(\ln x)}{x} x dx$$

$$= x \cos(\ln x) + \int \sin(\ln x) dx$$

$$= x \cos(\ln x) + \sin(\ln x) x - \int \frac{\cos(\ln x)}{x} x dx$$

$$= x \cos(\ln x) + \sin(\ln x) \cdot x - I$$

$$\Rightarrow 2I = x(\cos(\ln x) + \sin(\ln x)) + C$$

$$\Rightarrow I = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C$$

---

## Question 198

The integral  $\int \frac{3x^{13} + 2x^{11}}{(2x^4 + 3x^2 + 1)^4} dx$  is equal to:

(where C is a constant of integration)

[Jan.12,2019 (II)]

**Options:**

A.  $\frac{x^4}{6(2x^4 + 3x^2 + 1)^3} + C$

B.  $\frac{x^{12}}{6(2x^4 + 3x^2 + 1)^3} + C$

C.  $\frac{x^4}{(2x^4 + 3x^2 + 1)^3} + C$

D.  $\frac{x^{12}}{(2x^4 + 3x^2 + 1)^3} + C$

**Answer: B**

**Solution:**

$$I = \int \frac{3x^{13} + 2x^{11}}{(2x^4 + 3x^2 + 1)^4} dx = \int \frac{3x^{13} + 2x^{11}}{x^{16} \left(2 + \frac{3}{x^2} + \frac{1}{x^4}\right)^4} dx$$

$$I = \int \frac{\frac{3}{x^3} + \frac{2}{x^5}}{\left(2 + \frac{3}{x^2} + \frac{1}{x^4}\right)^4} dx$$

$$\text{Let } 2 + \frac{3}{x^2} + \frac{1}{x^4} = t, \quad -2\left(\frac{3}{x^3} + 2x^5\right) dx = dt$$

$$\text{Then, } I = \int \frac{-\frac{dt}{2}}{t^4} = -\frac{1}{2} \frac{t^{-4+1}}{-4+1} + C$$

$$I = \frac{-1}{2} \times \frac{1}{(-3)} \frac{1}{\left(2 + \frac{3}{x^2} + \frac{1}{x^4}\right)^3} + C$$

$$I = \frac{1}{6} \frac{x^{12}}{(2x^4 + 3x^2 + 1)^3} + C$$

## Question 199

If  $\int \frac{\sqrt{1-x^2}}{x^4} dx = A(x) \left(\sqrt{1-x^2}\right)^m + C$ , for a suitable chosen integer  $m$  and a function  $A(x)$ , where  $C$  is a constant of integration, then  $(A(x))^m$  equals:  
[Jan. 11, 2019 (I)]

Options:

A.  $\frac{-1}{27x^9}$

B.  $\frac{-1}{3x^3}$

C.  $\frac{1}{27x^6}$

D.  $\frac{1}{9x^4}$

**Answer: A**

**Solution:**



$$A(x)(\sqrt{1-x^2})^m + C = \int \frac{\sqrt{1-x^2}}{x^4} dx$$

$$= \int \frac{\sqrt{\frac{1}{x^2} - 1}}{x^3} dx$$

$$\text{Let } \frac{1}{x^2} - 1 = u^2$$

$$\Rightarrow -\frac{2}{x^3} = \frac{2ud u}{dx}$$

$$\frac{dx}{x^3} = -ud u$$

$$A(x)(\sqrt{1-x^2})^m + C = \int (-u^2) du = -\frac{u^3}{3} + C$$

$$= -\frac{1}{3} \left( \frac{1}{x^2} - 1 \right)^{\frac{3}{2}} + C$$

$$= -\frac{1}{3} \cdot \frac{1}{x^3} \cdot (1-x^2)^{\frac{3}{2}} + C$$

$$= \frac{-1}{3x^3} (\sqrt{1-x^2})^3 + C$$

Compare both sides,

$$\Rightarrow A(x) = -\frac{1}{3x^3} \text{ and } m = 3$$

$$\Rightarrow (A(x))^3 = \frac{-1}{27x^9}$$

## Question200

If  $\int \frac{x+1}{\sqrt{2x-1}} dx = f(x)\sqrt{2x-1} + C$ , where  $C$  is a constant of integration, then  $f(x)$  is equal to:

[Jan. 11, 2019 (II)]

Options:

A.  $\frac{1}{3}(x+1)$

B.  $\frac{2}{3}(x+2)$

C.  $\frac{2}{3}(x-4)$

D.  $\frac{1}{3}(x+4)$

**Answer: D**

## Solution:

$$\text{Let } I = \int \frac{x+1}{\sqrt{2x-1}} dx$$

$$\text{Put } \sqrt{2x-1} = t$$

$$\therefore 2x-1 = t^2 \Rightarrow dx = t dt$$

$$I = \int \frac{(t^2+3)}{2} dt = \frac{t^3}{6} + \frac{3t}{2} + C$$

$$= \frac{(2x-1)^{\frac{3}{2}}}{6} + \frac{3}{2}(2x-1)^{\frac{1}{2}} + C$$

$$= \sqrt{2x-1} \left( \frac{x+4}{3} \right) + C$$

$$= f(x) \cdot \sqrt{2x-1} + C$$

$$\text{Hence, } f(x) = \frac{x+4}{3}$$

---

## Question201

Let  $n \geq 2$  be a natural number and  $0 < \theta < \frac{\pi}{2}$

Then  $\int \frac{(\sin^n \theta + \sin \theta)^{\frac{1}{n}} \cos \theta}{\sin^{n+1} \theta} d\theta$  is equal to:

[Jan 10, 2019(I)]



**Options:**

$$\text{A. } \frac{n}{n^2-1} \left( 1 - \frac{1}{\sin^{n-1}\theta} \right)^{\frac{n+1}{n}} + C$$

$$\text{B. } \frac{n}{n^2+1} \left( 1 - \frac{1}{\sin^{n-1}\theta} \right)^{\frac{n+1}{n}} + C$$

$$\text{C. } \frac{n}{n^2-1} \left( 1 + \frac{1}{\sin^{n-1}\theta} \right)^{\frac{n+1}{n}} + C$$

$$\text{D. } \frac{n}{n^2-1} \left( 1 - \frac{1}{\sin^{n+1}\theta} \right)^{\frac{n+1}{n}} + C$$

**Answer: A**

**Solution:**

**Solution:**

$$\text{Let, } I = \int \frac{(\sin^n \theta - \sin \theta)^{\frac{1}{n}} \cos \theta}{\sin^{n+1} \theta} d\theta$$

$$\text{Let } \sin \theta = u$$

$$\Rightarrow \cos \theta d\theta = du$$

$$\begin{aligned} \therefore I &= \int \frac{(u^n - u)^{\frac{1}{n}}}{u^{n+1}} du \\ &= \int \frac{\left(1 - \frac{1}{u^{n-1}}\right)^{\frac{1}{n}}}{u^n} du = \int u^{-n} (1 - u^{1-n})^{\frac{1}{n}} du \\ \text{Let } 1 - u^{1-n} &= v \\ \Rightarrow -(1-n)u^{-n} du &= dv \\ \Rightarrow u^n du &= \frac{dv}{n-1} \\ \therefore I &= \int v^{\frac{1}{n}} \cdot \frac{dv}{n-1} = \frac{1}{n-1} \cdot \frac{v^{\frac{n+1}{n}} - 1}{\frac{n+1}{n}} \\ &= \frac{n+1}{n} v^{\frac{n+1}{n}} + C = \frac{n+1}{n} \left(1 - \frac{1}{u^{n-1}}\right)^{\frac{n+1}{n}} + C \\ &= \frac{n+1}{n} \left(1 - \frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}} + C \end{aligned}$$


---

## Question202

$\lim_{n \rightarrow \infty} \left( \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{1}{5n} \right)$  is equal to :

[Jan. 12, 2019 (II)]

Options:

A.  $\frac{\pi}{4}$

B.  $\tan^{-1}(3)$

C.  $\frac{\pi}{2}$

D.  $\tan^{-1}(2)$

**Answer: D**

**Solution:**

**Solution:**

$$\text{Let } L = \lim_{n \rightarrow \infty} \sum_{r=1}^{2n} \frac{n}{n^2 + r^2} = \int_0^2 \frac{dx}{1+x^2}$$

$$\left[ \because \frac{r}{n} \rightarrow x, \frac{1}{n} \rightarrow dx \right]$$

$$= [\tan^{-1}x]_0^2$$

$$= \tan^{-1}2$$

## Question203

Let  $f$  and  $g$  be continuous functions on  $[0, a]$  such that  $f(x) = f(a-x)$  and  $g(x) + g(a-x) = 4$ , then  $\int_0^a f(x)g(x)dx$  is equal to:

[Jan. 12, 2019 (I)]

**Options:**

A.  $4 \int_0^a f(x)dx$

B.  $\int_0^a f(x)dx$

C.  $2 \int_0^a f(x)dx$

D.  $-3 \int_0^a f(x)dx$

**Answer: C**

**Solution:**

**Solution:**

$$f(x) = f(a-x)$$

$$g(x) + g(a-x) = 4$$

Let, the integral,

$$I = \int_0^a f(x)g(x)dx$$

$$= \int_0^a f(a-x) \cdot g(a-x)dx$$

$$\left[ \because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx \right]$$

$$\Rightarrow I = \int_0^a f(x)[4 - g(x)]dx$$

$$\Rightarrow I = \int_0^a 4f(x)dx - \int_0^a f(x) \cdot g(x)dx$$

$$\Rightarrow I = \int_0^a 4f(x)dx - I$$

$$\Rightarrow 2I = \int_0^a 4f(x)dx$$

$$\Rightarrow I = 2 \int_0^a f(x)dx$$

## Question204

The integral  $\int_1^e \left\{ \left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^x \right\} \log_e x dx$  is equal to :

[Jan. 12, 2019 (II)]

Options:

A.  $\frac{1}{2} - e - \frac{1}{e^2}$

B.  $-\frac{1}{2} + \frac{1}{e} - \frac{1}{2e^2}$

C.  $\frac{3}{2} - \frac{1}{e} - \frac{1}{2e^2}$

D.  $\frac{3}{2} - e - \frac{1}{2e^2}$

**Answer: D**

**Solution:**

**Solution:**

$$I = \int_1^e \left\{ \left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^x \right\} \log_e x dx$$

$$\text{Let } \left(\frac{x}{e}\right)^x = t$$

$$\Rightarrow x \ln\left(\frac{x}{e}\right) = \ln t$$

$$\Rightarrow x(\ln x - 1) = \ln t$$

On differentiating both sides w.r. to  $x$  we get

$$\ln x \cdot dx = \frac{dt}{t}$$

When  $x = e$  then  $t = 1$  and when  $x = 1$  then  $t = \frac{1}{e}$ .

$$\begin{aligned} I &= \int_{\frac{1}{e}}^1 \left( t^2 - \frac{1}{t} \right) \cdot \frac{dt}{t} = \int_{\frac{1}{e}}^1 \left( t - \frac{1}{t^2} \right) dt \\ &= \left( \frac{t^2}{2} + \frac{1}{t} \right) \Big|_{\frac{1}{e}}^1 = \left( \frac{1}{2} + 1 \right) - \left( \frac{1}{2e^2} + e \right) = \frac{3}{2} - e - \frac{1}{2e^2} \end{aligned}$$

---

## Question 205

The value of the integral  $\int_{-2}^2 \frac{\sin^2 x}{\left[ \frac{x}{\pi} \right] + \frac{1}{2}} dx$

(where  $[x]$  denotes the greatest integer less than or equal to  $x$ ) is:  
[Jan. 11, 2019 (I)]

Options:

- A. 0
- B.  $\sin 4$
- C. 4
- D.  $4 - \sin 4$

**Answer: A**

**Solution:**

**Solution:**

$$\text{Let } f(x) = \frac{\sin^2 x}{\left[ \frac{x}{\pi} \right] + \frac{1}{2}}$$

$$\text{So, } f(-x) = \frac{\sin^2(-x)}{\left[ \frac{-x}{\pi} \right] + \frac{1}{2}} \because [-x] = -1 - [x]$$

$$\Rightarrow f(-x) = \frac{\sin^2 x}{-1 - \left[\frac{x}{\pi}\right] + \frac{1}{2}} = \frac{\sin^2 x}{-\frac{1}{2} - \left[\frac{x}{\pi}\right]} = -f(x)$$

$\Rightarrow f(x)$  is odd function

Hence,  $\int_{-2}^2 f(x) dx = 0$

---

## Question 206

The integral  $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)}$  equals:  
**[Jan. 11, 2019 (II)]**

**Options:**

A.  $\frac{1}{20} \tan^{-1} \left( \frac{1}{9\sqrt{3}} \right)$

B.  $\frac{1}{10} \left( \frac{\pi}{4} - \tan^{-1} \left( \frac{1}{9\sqrt{3}} \right) \right)$

C.  $\frac{\pi}{40}$

D.  $\frac{1}{5} \left( \frac{\pi}{4} - \tan^{-1} \left( \frac{1}{3\sqrt{3}} \right) \right)$

**Answer: B**

**Solution:**

**Solution:**

$$\begin{aligned} I &= \int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)} \\ &= \int_{\pi/6}^{\pi/4} \frac{\tan^5 x \cdot \sec^2 x}{2 \frac{\sin x}{\cos x} ((\tan^5 x)^2 + 1)} \\ &= \frac{1}{2} \int_{\pi/6}^{\pi/4} \frac{\tan^4 x \cdot \sec^2 x}{(\tan^5 x)^2 + 1} dx \end{aligned}$$

Let  $\tan^4 x = t$

$5 \tan^4 x \cdot \sec^2 x dx = dt$

When  $x \rightarrow \frac{\pi}{4}$  then  $t \rightarrow 1$

and  $x \rightarrow \frac{\pi}{6}$  then  $t \rightarrow \left(\frac{1}{\sqrt{3}}\right)^5$

$$\therefore I = \frac{1}{10} \int_{\left(\frac{1}{\sqrt{3}}\right)^5}^1 \frac{dt}{t^2+1}$$

$$= \frac{1}{10} \left( \frac{\pi}{4} - \tan^{-1} \left( \frac{1}{9\sqrt{3}} \right) \right)$$

---

## Question207

Let  $I = \int_a^b (x^4 - 2x^2) dx$ . If  $I$  is minimum then the ordered pair  $(a, b)$  is:  
[Jan 10, 2019 (I)]

**Options:**

A.  $(0, \sqrt{2})$

B.  $(-\sqrt{2}, 0)$

C.  $(\sqrt{2}, -\sqrt{2})$

D.  $(-\sqrt{2}, \sqrt{2})$

**Answer: D**

**Solution:**

**Solution:**

$$I = \int_a^b (x^4 - 2x^2) dx$$

$$\Rightarrow \frac{dI}{dx} = x^4 - 2x^2 = 0 \text{ (for minimum)}$$

$$\Rightarrow x = 0, \pm\sqrt{2}$$

$$\text{Also, } I = \left[ \frac{x^5}{5} - \frac{2x^3}{3} \right]_a^b$$

$$\text{For } a = 0, b = \sqrt{2}$$

$$I = \frac{-8\sqrt{2}}{15}$$

$$\text{For } a = -\sqrt{2}, b = 0$$

$$I = \frac{-8\sqrt{2}}{15}$$

$$\text{For } a = \sqrt{2}, b = -\sqrt{2}$$

$$I = \frac{16\sqrt{2}}{15}$$

$$\text{For } a = -\sqrt{2}, b = \sqrt{2}$$

$$I = \frac{-16\sqrt{2}}{15}$$

$\therefore I$  is minimum when  $(a, b) = (-\sqrt{2}, \sqrt{2})$

---

## Question208

If  $\int_0^x f(t) dt = x^2 + \int_x^1 t^2 f(t) dt$ , then  $f'(1/2)$  is:

[Jan. 10, 2019 (II)]

**Options:**

A.  $\frac{24}{25}$

B.  $\frac{18}{25}$

C.  $\frac{4}{5}$

D.  $\frac{6}{25}$

**Answer: A**

**Solution:**

**Solution:**

$$\int_0^x f(t) dt = x^2 + \int_x^1 t^2 f(t) dt$$

$$\Rightarrow f(x) = 2x - x^2 f(x)$$

$$\Rightarrow f(x) = \frac{2x}{1+x^2}$$

$$\Rightarrow f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}$$



Then,

$$f'(1/2) = \frac{2\left(1 - \frac{1}{4}\right)}{\left(1 + \frac{1}{4}\right)^2} = \frac{3}{2} \times \frac{16}{25} = \frac{24}{25}$$

---

## Question209

The value of  $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x] + [\sin x] + 4}$ , where  $[t]$  denotes the greatest integer less than or equal to  $t$ , is:  
[Jan. 10, 2019 (II)]

Options:

A.  $\frac{1}{12}(7\pi + 5)$

B.  $\frac{1}{12}(7\pi - 5)$

C.  $\frac{3}{20}(4\pi - 3)$

D.  $\frac{3}{10}(4\pi - 3)$

**Answer: C**

**Solution:**

**Solution:**

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \frac{dx}{[x] + [\sin x] + 4} \\ &= \int_{-\pi}^{-1} \frac{dx}{-2 - 1 + 4} + \int_{-1}^0 \frac{dx}{-1 - 1 + 4} + \int_0^1 \frac{dx}{0 + 0 + 4} + \int_1^{\frac{\pi}{2}} \frac{dx}{1 + 0 + 4} \\ &= \left(-1 + \frac{\pi}{2}\right) + \frac{1}{2}(0 + 1) + \frac{1}{4}(1 - 0) + \frac{1}{5}\left(\frac{\pi}{2} - 1\right) \\ &= \frac{3\pi}{5} - \frac{9}{20} = \frac{3}{20}(4\pi - 3) \end{aligned}$$

---

## Question210

The value of  $\int_0^{\pi} |\cos x|^3 dx$  is:

[Jan 9, 2019 (I)]

Options:

A. 0

B.  $\frac{4}{3}$

C.  $\frac{2}{3}$

D.  $-\frac{4}{3}$

Answer: B

Solution:

Solution:

$$\begin{aligned} I &= \int_0^{\pi} |\cos x|^3 dx \\ &= 2 \int_0^{\pi/2} \cos^3 x dx \\ &= \frac{2}{4} \int_0^{\pi/2} (3 \cos x + \cos 3x) dx \quad [\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta] \\ &= \frac{1}{2} \left[ 3 \sin x + \frac{\sin 3x}{3} \right]_0^{\pi/2} \\ &= \frac{1}{2} \left( 3 - \frac{1}{3} \right) = \frac{4}{3} \end{aligned}$$

---

## Question211

Let  $f$  be a differentiable function from  $\mathbb{R}$  to  $\mathbb{R}$  such that  $|f(x)$

$-f(y)| \leq 2|x - y|^{3/2}$ , for all  $x, y, \in \mathbb{R}$ . If  $f(0) = 1$  then  $\int_0^1 f^2(x) dx$  is equal

to :

[Jan. 09, 2019 (II)]

**Options:**

A. 1

B. 2

C.  $\frac{1}{2}$

D. 0

**Answer: A**

**Solution:**

**Solution:**

$\therefore f : \mathbb{R} \rightarrow \mathbb{R}$

and  $|f(x) - f(y)| \leq 2 \cdot |x - y|^{3/2}$

$$\Rightarrow \left| \frac{f(x) - f(y)}{x - y} \right| \leq 2\sqrt{x - y}$$

$$\Rightarrow \lim_{x \rightarrow y} \left| \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{x \rightarrow y} 2\sqrt{x - y}$$

$$\Rightarrow |f'(x)| = 0$$

$\therefore f(x)$  is a constant function.

Given  $f(0) = 1 \Rightarrow f(x) = 1$

Hence, the integral

$$\int_0^1 f^2(x) dx = \int_0^1 1 dx = [x]_0^1 = 1$$

---

## Question212

If  $\int_0^{\pi/3} \frac{\tan \theta}{\sqrt{2k \sec \theta}} d\theta = 1 - \frac{1}{\sqrt{2}}$ , ( $k > 0$ ) then the value of  $k$  is:

[Jan. 09, 2019 (II)]

**Options:**

A. 4

B.  $\frac{1}{2}$

C. 1

D. 2

**Answer: D**

**Solution:**

**Solution:**

$$\begin{aligned}\text{Let, } I &= \int_0^{\pi/3} \frac{\tan \theta}{\sqrt{2k \sec \theta}} d\theta \\ &= \frac{1}{\sqrt{2k}} \int_0^{\pi/3} \frac{\sin \theta}{\sqrt{\cos \theta}} d\theta\end{aligned}$$

$$\text{Let } \cos \theta = t^2$$

$$\therefore \sin \theta d\theta = -2tdt$$

Hence, integral becomes,

$$\begin{aligned}I &= \frac{1}{\sqrt{2k}} \int_1^{\frac{1}{\sqrt{2}}} \frac{-2tdt}{t} \\ &= \sqrt{\frac{2}{k}} \int_1^{\frac{1}{\sqrt{2}}} dt \\ &= \sqrt{\frac{2}{k}} \left(1 - \frac{1}{\sqrt{2}}\right) \\ &= \frac{\sqrt{2}-1}{\sqrt{k}} \\ &= 1 - \frac{1}{\sqrt{2}} \quad (\text{Given}) \\ \therefore k &= 2\end{aligned}$$

---

## Question213

If  $\int x^5 e^{-4x^3} dx = \frac{1}{48} e^{-4x^3} f(x) + C$ , where  $C$  is a constant of integration, then  $f(x)$  is equal to:  
[Jan. 10, 2019 (II)]

**Options:**

A.  $-2x^3 - 1$

B.  $-4x^3 - 1$

C.  $-2x^3 + 1$

D.  $4x^3 + 1$

**Answer: B**

**Solution:**

**Solution:**

$$I = \int x^5 e^{-4x^3} dx$$

$$\text{Put } -4x^3 = \theta$$

$$\Rightarrow -12x^2 dx = d\theta$$

$$\Rightarrow x^2 dx = -\frac{d\theta}{12}$$

$$I = \int \frac{1}{48} \theta e^\theta d\theta = \frac{1}{48} [\theta e^\theta - e^\theta] + C$$

$$I = \frac{1}{48} e^{-4x^3} (-4x^3 - 1) + C$$

Then, by comparison

$$f(x) = -4x^3 - 1$$

## Question214

A value of  $\alpha$  such that  $\int_{\alpha}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)} = \log_e \left( \frac{9}{8} \right)$  is:

[April 12, 2019 (II)]

**Options:**

A. -2

B.  $\frac{1}{2}$

C.  $-\frac{1}{2}$

D. 2

**Answer: A**

**Solution:**



$$\int_{\alpha}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)}$$

$$= \int_{\alpha}^{\alpha+1} \left[ \frac{1}{x+\alpha} - \frac{1}{x+\alpha+1} \right] dx \text{ [Using partial fraction]}$$

$$= \log \left( \frac{(x+\alpha)}{(x+\alpha+1)} \right) \Big|_{\alpha}^{\alpha+1} = \log \left( \frac{2\alpha+1}{2\alpha+2} \cdot \frac{2\alpha+1}{2\alpha} \right)$$

$$= \log \frac{9}{8} \text{ ( Given )}$$

$$\text{So, } \frac{(2\alpha+1)^2}{\alpha(\alpha+1)} = \frac{9}{2} \Rightarrow 8\alpha^2 + 8\alpha + 2 = 9\alpha^2 + 9\alpha$$

$$\Rightarrow \alpha^2 + \alpha - 2 = 0 \Rightarrow \alpha = 1, -2$$

## Question215

The integral  $\int \frac{2x^3-1}{x^4+x} dx$  is equal to :

(Here C is a constant of integration)

[April 12, 2019 (I)]

Options:

A.  $\frac{1}{2} \log_e \frac{|x^3+1|}{x^2} + C$

B.  $\frac{1}{2} \log_e \frac{(x^3+1)^2}{|x^3|} + C$

C.  $\log_e \left| \frac{x^3+1}{x} \right| + C$

D.  $\log_e \frac{|x^3+1|}{x^2} + C$

**Answer: C**

**Solution:**

**Solution:**

$$\text{Given integral, } I = \int \frac{(2x^3-1)dx}{x^4+x} = \int \frac{(2x-x^{-2})dx}{x^2+x^{-1}}$$

$$\text{Put } x^2+x^{-1} = u \Rightarrow (2x-x^{-2})dx = du$$

$$\Rightarrow I = \int \frac{du}{u} = \log |u| + c = \log |x^2 + x^{-1}| + c$$

$$= \log \left| \frac{x^3 + 1}{x} \right| + c$$


---

## Question216

Let  $\alpha \in (0, \pi/2)$  be fixed. If the integral  $\int \frac{\tan x + \tan \alpha}{\tan x - \tan \alpha} dx$   
 $= A(x) \cos 2\alpha + B(x) \sin 2\alpha + C$ , where  $C$  is a constant of integration,  
 then the functions  $A(x)$  and  $B(x)$  are respectively:  
 [April 12, 2019 (II)]

Options:

- A.  $x + \alpha$  and  $\log_e |\sin(x + \alpha)|$
- B.  $x - \alpha$  and  $\log_e |\sin(x - \alpha)|$
- C.  $x - \alpha$  and  $\log_e |\cos(x - \alpha)|$
- D.  $x + \alpha$  and  $\log_e |\sin(x - \alpha)|$

**Answer: B**

**Solution:**

**Solution:**

Given integral

$$\int \frac{\tan x + \tan \alpha}{\tan x - \tan \alpha} dx = \int \frac{\sin(x + \alpha)}{\sin(x - \alpha)} dx$$

Let  $x - \alpha = t \Rightarrow dx = dt$

$$= \int \frac{\sin(t + 2\alpha)}{\sin t} dt = \int [\cos 2\alpha + \sin 2\alpha \cdot \cot t] dt$$

$$= \cos 2\alpha \cdot t + \sin 2\alpha \cdot \log |\sin t| + c$$

$$= (x - \alpha) \cdot \cos 2\alpha + \sin 2\alpha \cdot \log |\sin(x - \alpha)| + c$$


---

## Question217

If  $\int \frac{dx}{(x^2 - 2x + 10)^2} = A \left( \tan^{-1} \left( \frac{x-1}{3} \right) + \frac{f(x)}{x^2 - 2x + 10} \right) + C$  where  $C$  is a constant of

integration, then :

[April 10, 2019 (I)]

Options:

A.  $A = \frac{1}{54}$  and  $f(x) = 3(x - 1)$

B.  $A = \frac{1}{81}$  and  $f(x) = 3(x - 1)$

C.  $A = \frac{1}{27}$  and  $f(x) = 9(x - 1)$

D.  $A = \frac{1}{54}$  and  $f(x) = 9(x - 1)^2$

**Answer: A**

**Solution:**

**Solution:**

$$\text{Let } I = \int \frac{dx}{(x^2 - 2x + 10)^2} = \int \frac{dx}{((x-1)^2 + 9)^2}$$

$$\text{Let } (x-1)^2 = 9 \tan^2 \theta \dots (i)$$

$$\Rightarrow \tan \theta = \frac{x-1}{3}$$

After differentiating equation ... (i), we get

$$2(x-1)dx = 18 \tan \theta \sec^2 \theta d\theta$$

$$\therefore I = \int \frac{18 \tan \theta \sec^2 \theta d\theta}{2 \times 3 \tan \theta \times 81 \sec^4 \theta}$$

$$I = \frac{1}{27} \int \cos^2 \theta d\theta = \frac{1}{27} \times \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$

$$I = \frac{1}{54} \left\{ \theta + \frac{\sin 2\theta}{2} \right\} + c$$

$$I = \frac{1}{54} \left[ \tan^{-1} \left( \frac{x-1}{3} \right) + \frac{1}{2} \times \frac{2 \left( \frac{x-1}{3} \right)}{1 + \left( \frac{x-1}{3} \right)^2} \right] + c$$

$$I = \frac{1}{54} \left[ \tan^{-1} \left( \frac{x-1}{3} \right) + \frac{3(x-1)}{x^2 - 2x + 10} \right] + c$$



Compare it with  $A \left[ \tan^{-1} \left( \frac{x-1}{b} \right) + \frac{f(x)}{x^2 - 2x + 10} \right] + c$ ,

we get:  $A = \frac{1}{54}$  and  $f(x) = 3(x-1)$

---

## Question218

If  $f(x)$  is a non-zero polynomial of degree four, having local extreme points at  $x = -1, 0, 1$ ; then the set  $S = \{x \in \mathbb{R} : f(x) = f(0)\}$  contains exactly:

[April 09, 2019 (I)]

Options:

- A. four irrational numbers.
- B. four rational numbers.
- C. two irrational and two rational numbers.
- D. two irrational and one rational number

**Answer: D**

**Solution:**

**Solution:**

Since, function  $f(x)$  have local extreem points at  $x = -1, 0, 1$ . Then

$$f(x) = K(x+1)x(x-1)$$

$$= K(x^3 - x)$$

$$\Rightarrow f(x) = K \left( \frac{x^4}{4} - \frac{x^2}{2} \right) + C \text{ (using integration)}$$

$$\Rightarrow f(0) = C$$

$$\because f(x) = f(0) \Rightarrow K \left( \frac{x^4}{4} - \frac{x^2}{2} \right) = 0$$

$$\Rightarrow \frac{x^2}{2} \left( \frac{x^2}{2} - 1 \right) = 0 \Rightarrow x = 0, 0, \sqrt{2}, -\sqrt{2}$$

$$\therefore S = \{0, -\sqrt{2}, \sqrt{2}\}$$

---



## Question219

The integral  $\int \sec^{2/3} x \operatorname{cosec}^{4/3} x dx$  is equal to:  
(Here C is a constant of integration)  
[April 09, 2019 (I)]

Options:

A.  $-3 \tan^{-1/3} x + C$

B.  $-\frac{3}{4} \tan^{-4/3} x + C$

C.  $-3 \cot^{-1/3} x + C$

D.  $3 \tan^{-1/3} x + C$

**Answer: A**

**Solution:**

**Solution:**

$$I = \int \sec^{\frac{2}{3}} x \cdot \operatorname{cosec}^{\frac{4}{3}} x dx$$

$$I = \int \frac{\sec^2 x dx}{\tan^{\frac{4}{3}} x}$$

Put  $\tan x = z$

$$\Rightarrow \sec^2 x dx = dz$$

$$\Rightarrow I = \int z^{-\frac{4}{3}} \cdot dz = \frac{z^{-\frac{1}{3}}}{\left(-\frac{1}{3}\right)} + C \Rightarrow I = -3(\tan x)^{-\frac{1}{3}} + C$$

---

## Question220

If  $\int e^{\sec x} (\sec x \tan x f(x) + (\sec x \tan x + \sec^2 x)) dx = e^{\sec x} f(x) + C$ , then a possible choice of  $f(x)$  is:  
[April 09, 2019(II)]

**Options:**

A.  $\sec x + \tan x + \frac{1}{2}$

B.  $\sec x - \tan x - \frac{1}{2}$

C.  $\sec x + x \tan x - \frac{1}{2}$

D.  $x \sec x + \tan x + \frac{1}{2}$

**Answer: A**

**Solution:**

**Solution:**

Given,

$$\int e^{\sec x} (\sec x \tan x f(x) + (\sec x \tan x + \sec^2 x)) dx = e^{\sec x} f(x) + C, \dots\dots(i)$$

$$\therefore \int e^{g(x)} ((g'(x)f(x)) + f'(x)) dx = e^{g(x)} \times f(x) + C$$

Our comparing above equation by equation (i),

$$f(x) = \int ((\sec x \tan x) + \sec^2 x) dx$$

$$\therefore f(x) = \sec x + \tan x + C$$

---

## Question221

$$\int \frac{\sin \frac{5x}{2}}{\sin \frac{x}{2}} dx \text{ is equal to :}$$

(where c is a constant of integration.)

[April 08, 2019 (I)]

**Options:**

A.  $x + 2 \sin x + 2 \sin 2x + c$

B.  $2x + \sin x + 2 \sin 2x + c$

C.  $x + 2 \sin x + \sin 2x + c$

D.  $2x + \sin x + \sin 2x + c$

**Answer: C**

**Solution:**

**Solution:**

$$\begin{aligned}\int \frac{\sin \frac{5x}{2}}{\sin \frac{x}{2}} dx &= \int \frac{2 \cos \frac{x}{2} \cdot \sin \frac{5x}{2}}{2 \cos \frac{x}{2} \cdot \sin \frac{x}{2}} dx \\ &= \int \frac{\sin 3x + \sin 2x}{\sin x} dx \\ &= \int (3 - 4\sin^2 x + 2 \cos x) dx \\ &[\because \sin 2x = 2 \sin x \cos x \text{ and } \sin 3x = 3 \sin x - 4\sin^3 x] \\ &= \int (3 - 2(1 - \cos 2x) + 2 \cos x) dx \\ &= \int (1 + 2 \cos x + 2 \cos 2x) dx \\ &= x + 2 \sin x + \sin 2x + c\end{aligned}$$

---

## Question222

If  $\int \frac{dx}{x^3(1+x^6)^{\frac{1}{3}}} = xf(x)(1+x^6)^{\frac{1}{3}} + C$ , where  $C$  is a constant of integration,

then the function  $f(x)$  is equal to :

[April 08,2019 (II)]

**Options:**

A.  $\frac{3}{x^2}$

B.  $-\frac{1}{6x^3}$

C.  $-\frac{1}{2x^2}$

D.  $-\frac{1}{2x^3}$

**Answer: D**

**Solution:**

**Solution:**

$$\text{Let, } \int \frac{dx}{x^3(1+x^6)^{\frac{2}{3}}} = \int \frac{dx}{x^7(1+x^{-6})^{\frac{2}{3}}}$$

$$\text{Put } 1+x^{-6} = t^3 \Rightarrow -6^{-7}dx = 3t^2 dt \Rightarrow \frac{dx}{x^7} = \left(-\frac{1}{2}\right)t^2 dt$$

$$\text{Now, } I = \int \left(-\frac{1}{2}\right) \frac{t^2 dt}{t^2} = -\frac{1}{2}t + C$$

$$= -\frac{1}{2}(1+x^{-6})^{\frac{1}{3}} + C = -\frac{1}{2} \frac{(1+x^6)^{\frac{1}{3}}}{x^2} + C$$

$$= -\frac{1}{2x^3}x(1+x^6)^{\frac{1}{3}} + C$$

$$\text{Hence, } f(x) = -\frac{1}{2x^3}$$

---

## Question223

If  $\int x^5 e^{-x^2} dx = g(x)e^{-x^2} + c$ , where  $c$  is a constant of integration, then  $g(-1)$  is equal to :  
[April 10,2019 (II)]

**Options:**

A. -1

B. 1

C.  $-\frac{5}{2}$

D.  $-\frac{1}{2}$

**Answer: C**

**Solution:**

**Solution:**

$$\text{Let, } I = \int x^2 \cdot e^{-x^2} dx$$

$$\text{Put } -x^2 = t \Rightarrow -2x dx = dt$$

$$1 = \int \frac{t^2 \cdot e^t dt}{(-2)} = \frac{-1}{2} e^t (t^2 - 2t + 2) + c$$

$$\therefore g(x) = \frac{-1}{2} (x^4 + 2x^2 + 2) \Rightarrow g(-1) = \frac{-5}{2}$$

---

## Question224

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuously differentiable function such that  $f(2) = 6$  and  $f'(2) = \frac{1}{48}$ .

If  $\int_6^{f(x)} 4t^3 dt = (x-2)g(x)$ , then  $\lim_{x \rightarrow 2} g(x)$  is equal to:

[April 12, 2019 (I)]

Options:

A. 18

B. 24

C. 12

D. 36

**Answer: A**

**Solution:**

**Solution:**

$$\text{Given, } \int_6^{f(x)} 4t^3 dt = (x-2)g(x)$$

Differentiating both sides,

$$4(f(x))^3 \cdot f'(x) = g'(x)(x-2) + g(x)$$

$$\text{Putting } x = 2, \frac{4(6)^3 \cdot 1}{48} = g(2) \Rightarrow \lim_{x \rightarrow 2} g(x) = 18$$

---

## Question225

If  $\int_0^{\frac{\pi}{2}} \frac{\cot x}{\cot x + \operatorname{cosec} x} dx = m(\pi + n)$ , then  $m \cdot n$  is equal to :

[April 12, 2019 (I)]

Options:

A.  $-\frac{1}{2}$

B. 1

C.  $\frac{1}{2}$

D. -1

Answer: D

Solution:

Solution:

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \frac{\cot x}{\cot x + \operatorname{cosec} x} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\cot x dx}{1 + \cos x} = \int_0^{\frac{\pi}{2}} \left( 1 - \frac{1}{1 + \cos x} \right) dx \\ &= [x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1}{2 \cos^2 \frac{x}{2}} dx = \frac{\pi}{2} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2} dx \\ &= \frac{\pi}{2} - \left( \tan \frac{x}{2} \right)_0^{\pi/2} = \frac{\pi}{2} - [1] = \left( \frac{\pi}{2} - 1 \right) = m\pi + mn \\ &\therefore m = , n = -2, \text{ Hence, } mn = -1 \end{aligned}$$

---

## Question226

The value of  $\int_0^{2\pi} [\sin 2x(1 + \cos 3x)] dx$ , where  $[t]$  denotes the greatest integer function, is:

[April 10, 2019 (I)]

Options:

- A.  $\pi$
- B.  $-\pi$
- C.  $-2\pi$
- D.  $2\pi$

**Answer: B**

**Solution:**

**Solution:**

$$I = \int_0^{2\pi} [\sin 2x(1 + \cos 3x)]dx \dots\dots(1)$$

$$\because \int_0^a f(x) = \int_0^a f(a-x)dx$$

$$\therefore I = \int_0^{2\pi} [-\sin 2x(1 + \cos 3x)]dx \dots\dots(2)$$

From (1) + (2), we get;

$$2I = \int_0^{2\pi} (-1)dx \Rightarrow 2I = -(x)_0^{2\pi} \Rightarrow I = -\pi$$

## Question227

The integral  $\int_{\pi/6}^{\pi/3} \sec^{2/3} x \cos e c^{4/3} x dx$  is equal to:  
[April 10, 2019(II)]

**Options:**

- A.  $3^{5/6} - 3^{2/3}$
- B.  $3^{4/3} - 3^{1/3}$
- C.  $3^{7/6} - 3^{5/6}$
- D.  $3^{5/3} - 3^{1/3}$

**Answer: C**

**Solution:**



$$\text{Let, } I = \int_{\pi/6}^{\pi/3} \sec^{\frac{2}{3}}x \cdot \operatorname{cosec}^{\frac{4}{3}}x dx = \int_{\pi/6}^{\pi/3} \frac{1 \cdot dx}{\cos^{\frac{2}{3}}x \cdot \sin^{\frac{4}{3}}x}$$

$$= \int_{\pi/6}^{\pi/3} \frac{1 dx}{\cos^2x \cdot \tan^{\frac{4}{3}}x} = \int_{\pi/6}^{\pi/3} \frac{\sec^2x dx}{\tan^{\frac{4}{3}}x}$$

Let  $\tan x = u$

$$I = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} u^{-\frac{4}{3}} du = \frac{3 \left[ u^{-\frac{1}{3}} \right]}{\frac{1}{\sqrt{3}}}$$

$$= -3 \left[ 3^{-\frac{1}{6}} - \frac{1}{3^{-\frac{1}{6}}} \right] = -3 \left( 3^{-\frac{1}{6}} - 3^{\frac{1}{6}} \right)$$

$$= 3 \left( 3^{\frac{1}{6}} - 3^{-\frac{1}{6}} \right) = \left( 3^{\frac{7}{6}} - 3^{\frac{5}{6}} \right)$$


---

## Question 228

The value of  $\int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x} dx$  is:

[April 9, 2019 (I)]

Options:

A.  $\frac{\pi-2}{8}$

B.  $\frac{\pi-1}{4}$

C.  $\frac{\pi-2}{4}$

D.  $\frac{\pi-1}{2}$

**Answer: B**

**Solution:**

$$\text{Let } I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x} dx \dots\dots(1)$$

Use the property  $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$I = \int_0^{\pi/2} \frac{\cos^3 x dx}{\sin x + \cos x} \dots\dots(2)$$

Adding equation (1) and (2), we get

$$\Rightarrow 2I = \int_0^{\pi/2} \left(1 - \frac{1}{2} \sin(2x)\right) dx$$

$$\Rightarrow I = \frac{1}{2} \left[ x + \frac{1}{4} \cos 2x \right]_0^{\pi/2}$$

$$\Rightarrow I = \frac{\pi - 1}{4}$$

## Question229

The value of the integral  $\int_0^1 x \cot^{-1}(1 - x^2 + x^4) dx$  is:

[April 09, 2019 (II)]

Options:

A.  $\frac{\pi}{2} - \frac{1}{2} \log_e 2$

B.  $\frac{\pi}{4} - \log_e 2$

C.  $\frac{\pi}{2} - \log_e 2$

D.  $\frac{\pi}{4} - \frac{1}{2} \log_e 2$

**Answer: D**

**Solution:**

**Solution:**

$$\int_0^1 x \cot^{-1}(1 - x^2 + x^4) dx = \int_0^1 x \tan^{-1} \left( \frac{1}{1 + x^4 - x^2} \right)$$

$$= \int_0^1 x \tan^{-1} \left( \frac{x^2 - (x^2 - 1)}{1 + x^2(x^2 - 1)} \right) dx$$

$$= \frac{1}{2} \int_0^1 1 \tan^{-1} t^2 dt - \frac{1}{2} \int_{-1}^0 1 \tan^{-1} k dk$$

Put  $x^2 = t \Rightarrow 2x dx = dt$  in the first integral and  
 $x^2 - 1 = k \Rightarrow 2x dx = dk$  in the second integral.

$$\begin{aligned}
 &= \frac{1}{2} \int_0^1 \tan^{-1} t dt - \frac{1}{2} \int_0^1 \tan^{-1} k dk \\
 &= \frac{1}{2} \left( t \tan^{-1} t \Big|_0^1 - \int_0^1 \frac{t}{1+t^2} dt \right) - \frac{1}{2} \left( k \tan^{-1} k \Big|_0^1 - \int_{-1}^0 \frac{k}{1+k^2} dk \right) \\
 &= \frac{1}{2} \left( \frac{\pi}{4} - \left( \frac{1}{2} \ln(1+t^2) \Big|_0^1 \right) \right) - \frac{1}{2} \left( -\frac{\pi}{4} - \left( \frac{1}{2} \ln(1+k^2) \Big|_{-1}^0 \right) \right) \\
 &= \left( \frac{\pi}{8} - \frac{1}{4} \ln 2 \right) - \left( -\frac{\pi}{8} - \frac{1}{4} \ln 2 \right) = \frac{\pi}{4} - \frac{1}{2} \ln 2
 \end{aligned}$$


---

## Question 230

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable function and  $f(2) = 6$ , then  $\lim_{x \rightarrow 2} \int_6^{f(x)} \frac{2tdt}{(x-2)}$

is:

[April 09, 2019 (II)]

Options:

A.  $24f'(2)$

B.  $2f'(2)$

C. 0

D.  $12f'(2)$

**Answer: D**

**Solution:**

**Solution:**

Using L' Hospital rule and Leibnitz theorem, we get

$$\lim_{x \rightarrow 2} \frac{\int_6^{f(x)} 2tdt}{(x-2)} = \lim_{x \rightarrow 2} \frac{2f(x)f'(x) - 0}{1}$$

Putting  $x = 2$ ,  $2f(2)f'(2) = 12f'(2)$  [ $\because f(2) = 6$ ]

---

## Question 231

If  $f(x) = \frac{2 - x \cos x}{2 + x \cos x}$  and  $g(x) = \log_e x$ , ( $x > 0$ ) then the value of the integral

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} g(f(x)) dx \text{ is :}$$

[April 8, 2019 (I)]

Options:

A.  $\log_e 2$

B.  $\log_e 3$

C.  $\log_e e$

D.  $\log_e 1$

Answer: D

Solution:

Solution:

$$g(f(x)) = \log\left(\frac{2 - x \cos x}{2 + x \cos x}\right), x > 0$$

$$\text{Let } I = \int_{-\pi/4}^{\pi/4} \log\left(\frac{2 - x \cos x}{2 + x \cos x}\right) dx \dots\dots\dots(i)$$

Use the property  $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

Then, equation (i) becomes,

$$I = \int_{-\pi/4}^{\pi/4} \log\left(\frac{2 + x \cos x}{2 - x \cos x}\right) dx \dots\dots\dots(ii)$$

Adding (i) and (ii)

$$2I = \int_{-\pi/4}^{\pi/4} \log\left(\frac{2 - x \cos x}{2 + x \cos x} \cdot \frac{2 + x \cos x}{2 - x \cos x}\right) dx$$

$$2I = \int_{-\pi/4}^{\pi/4} \log(1) dx = 0$$

$$\Rightarrow I = 0 = \log 1$$

---

## Question232

Let  $f(x) = \int_0^x g(t) dt$ , where  $g$  is a non-zero even function.

If  $f(x+5) = g(x)$ , then  $\int_0^x f(t) dt$  equals :

[April 08, 2019(II)]

Options:

A.  $\int_{x+5}^5 g(t) dt$

B.  $\int_5^{x+5} g(t) dt$

C.  $2 \int_5^{x+5} g(t) dt$

D.  $5 \int_{x+5}^5 g(t) dt$

**Answer: A**

**Solution:**

**Solution:**

$$f(x) = \int_0^x g(t) dt, \dots\dots(i)$$

$\therefore g$  is a non-zero even function.

$$\therefore g(-x) = g(x) \dots\dots(ii)$$

$$\text{Given, } f(x+5) = g(x) \dots\dots(iii)$$

$$\text{From (i) } f'(x) = g(x)$$

$$\text{Let, } I = \int_0^x f(t) dt,$$

$$\text{Put } t = \lambda - 5 \Rightarrow I = \int_5^{x+5} f(\lambda - 5) d\lambda$$

$$\therefore f(x+5) = g(x)$$

$$\Rightarrow f(-x+5) = g(-x) = g(x) \dots\dots(iv)$$

$$I = \int_5^{x+5} f(\lambda - 5) d\lambda$$

$$f(0) = 0, g(x) \text{ is even} \Rightarrow f(x) \text{ is odd}$$

$$\therefore I = \int_5^{x+5} -f(5 - \lambda) d\lambda$$

$$\Rightarrow I = \int_5^{x+5} g(\lambda) d\lambda = \int_{x+5}^5 g(t) dt \text{ (from$$

## Question233

$\lim_{n \rightarrow \infty} \left( \frac{(n+1)^{1/3}}{n^{4/3}} + \frac{(n+2)^{1/3}}{n^{4/3}} + \dots + \frac{(2n)^{1/3}}{n^{4/3}} \right)$  is equal to :

[April 10, 2019 (I)]

Options:

A.  $\frac{3}{4}(2)^{4/3} - \frac{3}{4}$

B.  $\frac{4}{3}(2)^{4/3}$

C.  $\frac{3}{2}(2)^{4/3} - \frac{4}{3}$

D.  $\frac{4}{3}(2)^{3/4}$

Answer: A

Solution:

Solution:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left( \frac{(n+1)^{1/3}}{n^{4/3}} + \frac{(n+2)^{1/3}}{n^{4/3}} + \dots + \frac{(2n)^{1/3}}{n^{4/3}} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{(n+r)^{1/3}}{n \cdot n^{4/3}} \\ &= \int_0^1 (1+x)^{1/3} dx \quad \left[ \because \frac{r}{n} \rightarrow x \text{ and } \frac{1}{n} \rightarrow \frac{dx}{x} \right] \\ &= \left[ \frac{3}{4}(1+x)^{4/3} \right]_0^1 = \frac{3}{4}(2)^{4/3} - \frac{3}{4} \end{aligned}$$

---

## Question234

The integral

$$\int \frac{\sin^2 x \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2} dx$$

is equal to:  
[2018]

Options:

A.  $\frac{-1}{3(1 + \tan^3 x)} + C$

B.  $\frac{1}{1 + \cot^3 x} + C$

C.  $\frac{-1}{1 + \cot^3 x} + C$

D.  $\frac{1}{3(1 + \tan^3 x)} + C$

**Answer: A**

**Solution:**

**Solution:**

Let I

$$\begin{aligned} & \int \frac{\sin^2 x \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2} dx \\ &= \int \frac{\sin^2 x \cos^2 x}{[(\sin^2 x + \cos^2 x)(\sin^3 x + \cos^3 x)]^2} dx \\ &= \int \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx = \int \frac{\tan^2 x \cdot \sec^2 x}{(1 + \tan^3 x)^2} dx \end{aligned}$$

Now, put  $(1 + \tan^3 x) = t$

$$\Rightarrow 3 \tan^2 x \sec^2 x dx = dt$$

$$\therefore I = \frac{1}{3} \int \frac{dt}{t^2} = -\frac{1}{3t} + C = \frac{-1}{3(1 + \tan^3 x)} + C$$

---

## Question 235

$$\text{If } \int \frac{\tan x}{1 + \tan x + \tan^2 x} dx = x - \frac{K}{\sqrt{A}} \tan^{-1} \left( \frac{K \tan x + 1}{\sqrt{A}} \right) + C,$$

(C is a constant of integration), then the ordered pair (K, A) is equal to

[Online April 16, 2018]

**Options:**

A. (2,3)

B. (2,1)

C. (-2,1)

D. (-2,3)

**Answer: A**

**Solution:**

**Solution:**

$$\text{Let } I = \int \frac{\tan x}{1 + \tan x + \tan^2 x} dx$$

$$\Rightarrow I = \int \frac{\tan x + 1 + \tan^2 x}{\tan x + 1 + \tan^2 x} dx - \int \frac{(1 + \tan^2 x)}{1 + \tan x + \tan^2 x}$$

$$\Rightarrow I = x - \int \frac{\sec^2 x dx}{1 + \tan x + \tan^2 x}$$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x \cdot dx = dt$$

$$\therefore I = x - \int \frac{dt}{t^2 + t + \frac{1}{4} + 1 - \frac{1}{4}}$$

$$= x - \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow I = x - \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C$$

$$\Rightarrow I = x - \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2 \tan x + 1}{\sqrt{3}} \right) + C$$

$$\therefore A = 3 \text{ and } K = 2$$

---

## Question236

If  $f\left(\frac{x-4}{x+2}\right) = 2x + 1$ , ( $x \in \mathbb{R} = \{1, -2\}$ ), then  $\int f(x) dx$  is equal to

(where  $C$  is a constant of integration)

[Online April 15, 2018]





**Options:**

A.  $12\log_e |1 - x| - 3x + c$

B.  $-12\log_e |1 - x| - 3x + c$

C.  $-12\log_e |1 - x| + 3x + c$

D.  $12\log_e |1 - x| + 3x + c$

**Answer: B**

**Solution:**

**Solution:**

Suppose,  $\frac{x-4}{x+2} = y \Rightarrow x-4 = y(x+2)$

$\Rightarrow x(1-y) = 2y+4 \Rightarrow x = \frac{2y+4}{1-y}$

So,  $f(y) = 2\left(\frac{2y+4}{1-y}\right) + 1$

Now,  $f(x) = 2\left(\frac{2x+4}{1-x}\right) + 1 = \frac{3x+9}{1-x}$   
 $= \frac{3(x+3)}{1-x} = \frac{3(x-1+4)}{1-x} = -3 + \frac{12}{1-x}$

$\therefore \int f(x) dx = -12\log_e |1-x| - 3x + c$

---

## Question 237

$$\int \frac{2x+5}{\sqrt{7-6x-x^2}} dx = A\sqrt{7-6x-x^2} + B\sin^{-1}\left(\frac{x+3}{4}\right) + C$$

(where C is a constant of integration), then the ordered pair (A, B) is equal to

[Online April 15, 2018]

**Options:**

A. (-2,-1)

B. (2,-1)

C. (-2,1)

D. (2,1)

**Answer: A**

**Solution:**

**Solution:**

$$\because 7 - 6x - x^2 = 16 - (x + 3)^2$$

$$\text{and } \frac{d}{dx}(7 - 6x - x^2) = -2x - 6$$

$$\text{So, } \int \frac{2x + 5}{\sqrt{7 - 6x - x^2}} dx = \int \frac{2x + 6}{\sqrt{7 - 6x - x^2}} dx$$

$$- \int \frac{1}{\sqrt{16 - (x + 3)^2}} dx$$

$$= -2 \sqrt{7 - 6x - x^2} - \sin^{-1} \left( \frac{x + 3}{4} \right) + C$$

Therefore,  $A = -2$ , &  $B = -1$

---

## Question238

The value of  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1 + 2^x} dx$  is

[2018]

**Options:**

A.  $\frac{\pi}{2}$

B.  $4\pi$

C.  $\frac{\pi}{4}$

D.  $\frac{\pi}{8}$

**Answer: C**

## Solution:

### Solution:

$$\text{Let, } I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx \dots\dots(i)$$

Using  $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$ , we get :

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}} dx \dots\dots(ii)$$

Adding (i) and (ii), we get;

$$2I = \int_{-\pi/2}^{\pi/2} \sin^2 x dx \Rightarrow 2I = 2 \cdot \int_0^{\pi/2} \sin^2 x dx$$

$$\Rightarrow 2I = 2 \times \frac{\pi}{4} \Rightarrow I = \frac{\pi}{4}$$

---

## Question239

If  $f(x) = \int_0^x t(\sin x - \sin t) dt$  then  
[Online April 16, 2018]

### Options:

A.  $f'''(x) + f'(x) = \cos x - 2x \sin x$

B.  $f'''(x) + f''(x) - f'(x) = \cos x$

C.  $f'''(x) - f''(x) = \cos x - 2x \sin x$

D.  $f'''(x) + f''(x) = \sin x$

**Answer: A**

## Solution:

### Solution:

$$\begin{aligned} f(x) &= \int_0^x t(\sin x - \sin t) \cdot dt \\ &= \sin x \int_0^x t \cdot dt - \int_0^x t \sin t \cdot dt \\ &= \frac{x^2}{2} \sin x + [t \cos t]_0^x + \sin x \end{aligned}$$

$$\Rightarrow f(x) = \frac{x^2}{2} \sin x + x \cos x - \sin x$$

$$f'(x) = \frac{x^2}{2} \cos x + 2 \cos x$$

$$f''(x) = x \cos x - \frac{x^2}{2} \sin x - 2 \sin x$$

$$f'''(x) = \cos x - 2x \sin x - \frac{x^2}{2} \cos x - 2 \cos x$$

$$\therefore f'''(x) + f'(x) = \cos x - 2x \sin x$$

---

## Question240

The value of integral  $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{x}{1 + \sin x} dx$  is

[Online April 15, 2018]

Options:

A.  $\frac{\pi}{2}(\sqrt{2} + 1)$

B.  $\pi(\sqrt{2} - 1)$

C.  $2\pi(\sqrt{2} - 1)$

D.  $\pi\sqrt{2}$

**Answer: A**

**Solution:**

**Solution:**

$$\text{Let } I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{x}{1 + \sin x} dx$$

$$\text{also let } K = \frac{x}{1 + \sin x}$$

Multiplying numerator and denominator by  $(1 - \sin x)$ , we get;

$$K = \frac{x(1 - \sin x)}{1 - (\sin x)^2} = \frac{x(1 - \sin x)}{(\cos x)^2}$$

$$= x(1 - \sin x)\sec^2 x$$

$$= x\sec^2 x - x \sin x \sec^2 x = x\sec^2 x - x \tan x \sec x$$

$$\text{Now, } I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\sec^2 x dx - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x \sec x \tan x dx$$

$$= \left[ x \tan x - \int \frac{dx}{dx} \tan x dx \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} - \left[ x \sec x - \int \frac{dx}{dx} \sec x dx \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= [x \tan x - \ln |\sec x|]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$- [x \sec x - \ln |\sec x + \tan x|]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + c$$

$$\Rightarrow I = \left\{ \left[ \frac{3\pi}{4} \tan \frac{3\pi}{4} - \ln \left| \frac{3\pi}{4} \right| \right] - \left[ \frac{3\pi}{4} \sec \frac{3\pi}{4} - \ln \left| \sec \frac{3\pi}{4} + \tan \frac{3\pi}{4} \right| \right] \right\} - \left\{ \left[ \frac{\pi}{4} \tan \frac{\pi}{4} - \ln \left| \frac{\pi}{4} \right| \right] - \left[ \frac{\pi}{4} \sec \frac{\pi}{4} - \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| \right] \right\}$$

$$= \frac{\pi}{2}(\sqrt{2} + 1)$$

## Question 241

If  $I_1 = \int_0^1 e^{-x} \cos^2 x dx$ ;  $I_2 = \int_0^1 e^{-x^2} \cos^2 x dx$  and  $I_3 = \int_0^1 e^{-x^3} dx$ ; then  
**[Online April 15, 2018]**

**Options:**

A.  $I_2 > I_3 > I_1$

B.  $I_3 > I_1 > I_2$

C.  $I_2 > I_1 > I_3$

D.  $I_3 > I_2 > I_1$

**Answer: D**

**Solution:**

$$\text{Given: } I_1 = \int_0^1 e^{-x} \cos^2 x \, dx;$$

$$I_2 = \int_0^1 e^{-x^2} \cos^2 x \, dx$$

$$I_3 = \int_0^1 e^{-x^3} \, dx;$$

For  $x \in (0, 1)$

$$\Rightarrow x > x^2 \text{ or } -x < -x^2$$

$$\text{and } x^2 > x^3 \text{ or } -x^2 < -x^3$$

$$\therefore e^{-x^2} < e^{-x^3} \text{ and } e^{-x} < e^{-x^2}$$

$$\Rightarrow e^{-x} < e^{-x^2} < e^{-x^3}$$

$$\Rightarrow e^{-x^3} > e^{-x^2} > e^{-x}$$

$$\Rightarrow I_3 > I_2 > I_1$$

---

## Question242

The value of the integral  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \left( 1 + \log \left( \frac{2 + \sin x}{2 - \sin x} \right) \right) \, dx$  is

[Online April 15, 2018]

Options:

A.  $\frac{3}{16}\pi$

B. 0

C.  $\frac{3}{8}\pi$

D.  $\frac{3}{4}$

**Answer: C**

**Solution:**

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \left( 1 + \log \left( \frac{2 + \sin x}{2 - \sin x} \right) \right) dx \dots\dots(1)$$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4(-x) \left( 1 + \log \left( \frac{2 + \sin x}{2 - \sin x} \right) \right) \cdot dx$$

$$= \left[ \because \int_a^b f(x) \cdot dx = \int_a^b f(a + b - x) \cdot dx \right]$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^4 x) \left( 1 + \log \left( \frac{2 - \sin x}{2 + \sin x} \right) \right) \cdot dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \left( 1 - \log \left( \frac{2 + \sin x}{2 - \sin x} \right) \right) \cdot dx \dots\dots(2)$$

After adding equation (1) and (2) we get,

$$2I = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \cdot dx$$

$$2I = 4 \int_0^{\frac{\pi}{2}} \sin^4 x \cdot dx$$

$$I = 2 \int_0^{\frac{\pi}{2}} \sin^4 x \cdot dx = \frac{2 \times \frac{3}{2} \times \frac{1}{2} \times \pi}{2 \times 2} = \frac{3\pi}{8}$$

[By Gamma function]

## Question243

If  $f \left( \frac{3x-4}{3x+4} \right) = x + 2$ ,  $x \neq -\frac{4}{3}$ , and  $\int f(x) dx = A \log |1-x| + Bx + C$ , then the ordered pair(A, B) is equal to:  
(where C is a constant of integration)  
[Online April 9, 2017]

Options:

A.  $\left(\frac{8}{3}, \frac{2}{3}\right)$

B.  $\left(-\frac{8}{3}, \frac{2}{3}\right)$

C.  $\left(-\frac{8}{3}, -\frac{2}{3}\right)$

D.  $\left(\frac{8}{3}, -\frac{2}{3}\right)$

**Answer: B**

**Solution:**

**Solution:**

$$f\left(\frac{3x-4}{3x+4}\right) = x + 2, x \neq -\frac{4}{3}$$

Consider  $\frac{3x-4}{3x+4} = t$

$$\Rightarrow 3x - 4 = 3tx + 4t$$

$$\Rightarrow x = \frac{4t+4}{3-3t} + 2$$

$$\Rightarrow f(t) = \frac{10-2t}{3-3t}$$

$$\Rightarrow f(x) = \frac{2x-10}{3x-3}$$

$$\therefore \int f(x) dx = \int \frac{2x-10}{3x-3} dx$$

$$= \int \frac{2x}{3x-3} dx - 10 \int \frac{dx}{3x-3}$$

$$= \frac{2}{3} \int \frac{x-1}{x-1} dx + \frac{2}{3} \int \frac{dx}{x-1} - \frac{10}{3} \int \frac{dx}{x-1}$$

$$= \frac{2x}{3} - \frac{8}{3} \ln(x-1) + C$$

Here,  $A = -\frac{8}{3}, B = \frac{2}{3}$

$$\therefore (A, B) = \left(-\frac{8}{3}, \frac{2}{3}\right)$$

---

## Question244



The integral  $\int \sqrt{1 + 2 \cot x(\operatorname{cosec} x + \cot x)} dx \left( 0 < x < \frac{\pi}{2} \right)$  is equal to:

(where C is a constant of integration)

[Online April 8, 2017]

Options:

A.  $2 \log \left| \sin \frac{x}{2} \right| + C$

B.  $4 \log \left| \sin \frac{x}{2} \right| + C$

C.  $2 \log \left| \cos \frac{x}{2} \right| + C$

D.  $4 \log \left| \cos \frac{x}{2} \right| + C$

**Answer: A**

**Solution:**

**Solution:**

$$\text{Let, } I = \int \sqrt{1 + 2 \cot x \operatorname{cosec} x + 2 \cot^2 x} \cdot dx$$

$$\Rightarrow I = \int \sqrt{\frac{\sin^2 x + 2 \cos x + 2 \cos^2 x}{\sin^2 x}} \cdot dx$$

$$\Rightarrow I = \int \sqrt{\frac{1 + 2 \cos x + \cos^2 x}{\sin x}} dx$$

$$\Rightarrow I = \int \left| \frac{1 + \cos x}{\sin x} \right| \cdot dx$$

$$\Rightarrow I = \int |\operatorname{cosec} x + \cot x| \cdot dx$$

$$\Rightarrow I = \log |\operatorname{cosec} x - \cot x| + \log |\sin x| + C$$

$$\Rightarrow I = \log |1 - \cos x| + C$$

$$\Rightarrow I = \log \left| 2 \sin^2 \frac{x}{2} \right| + C$$

$$\Rightarrow I = \log \left| \sin^2 \frac{x}{2} \right| + \log 2 + C$$

$$\Rightarrow I = 2 \log \left| \sin \frac{x}{2} \right| + C_1$$

## Question245

The integral  $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1 + \cos x}$  is equal to :

[2017]

Options:

A. -1

B. -2

C. 2

D. 4

Answer: C

Solution:

Solution:

$$I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1 + \cos x} \dots\dots(i)$$

$$I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1 - \cos x} \dots\dots(ii)$$

$$\text{Using } \int_a^b f(x) dx = \int_a^b f(a + b - x) dx$$

Adding (i) and (ii)

$$2I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{2}{\sin^2 x} dx; I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \operatorname{cosec}^2 x dx$$

$$I = -(\cot x)_{\pi/4}^{3\pi/4} = -\left[ \cot \frac{3\pi}{4} - \cot \frac{\pi}{4} \right] = 2$$



## Question246

Let  $I_n = \int \tan^n x dx$ , ( $n > 1$ ).  $I_4 + I_6 = a \tan^5 x + bx^5 + C$

where  $C$  is constant of integration, then the ordered pair  $(a, b)$  is equal to :

[2017]

Options:

A.  $\left(-\frac{1}{5}, 0\right)$

B.  $\left(-\frac{1}{5}, 1\right)$

C.  $\left(\frac{1}{5}, 0\right)$

D.  $\left(\frac{1}{5}, -1\right)$

Answer: C

Solution:

Solution:

$$I_n = \int \tan^n x dx, n > 1$$

$$\text{Let } I = I_4 + I_6$$

$$= \int (\tan^4 x + \tan^6 x) dx = \int \tan^4 x \sec^2 x dx$$

$$\text{Let } \tan x = t$$

$$\Rightarrow \sec^2 x dx = dt$$

$$\therefore I = \int t^4 dt = \frac{t^5}{5} + C$$

$$= \frac{1}{5} \tan^5 x + C \Rightarrow \text{On comparing, we have}$$

$$a = \frac{1}{5}, b = 0$$

---

## Question247

If  $\int_1^2 \frac{dx}{(x^2 - 2x + 4)^{3/2}} = \frac{k}{k+5}$  then k is equal to :

[Online April 9, 2017]

Options:

- A. 1
- B. 2
- C. 3
- D. 4

Answer: A

Solution:

Solution:

$$\text{Let } I = \int_1^2 \frac{dx}{((x-1)^2 + 3)^{3/2}}$$

$$\text{Let; } x - 1 = \sqrt{3} \tan \theta$$

$$\Rightarrow dx = \sqrt{3} \sec^2 \theta \cdot d\theta$$

$$\Rightarrow I = \int_0^{\pi/6} \frac{\sqrt{3} \sec^2 \theta d\theta}{((\sqrt{3} \tan \theta)^2 + (\sqrt{3})^2)^{3/2}}$$

$$= \frac{1}{3} \int_0^{\pi/6} \frac{\sec^2 \theta}{\sec^3 \theta} d\theta = \frac{1}{3} \int_0^{\pi/6} \cos \theta d\theta$$

$$= \frac{1}{3} [\sin \theta]_0^{\pi/6} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$= \frac{1}{6} = \frac{k}{k+5} \Rightarrow k+5 = 6k$$

$$\Rightarrow k = 1$$

---

## Question248

The integral  $\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8 \cos 2x}{(\tan x + \cot x)^3} dx$  equals:

[Online April 8, 2017]

**Options:**

A.  $\frac{15}{128}$

B.  $\frac{15}{64}$

C.  $\frac{13}{32}$

D.  $\frac{15}{256}$

**Answer: A**

**Solution:**

**Solution:**

$$\begin{aligned} \frac{\pi}{4} \int \frac{\cos 2x}{\left(\frac{1}{\sin 2x}\right)^3} dx &= \frac{\pi}{4} \int \cos 2x \times \sin 2x \cdot \sin^2(2x) dx \\ \frac{\pi}{12} \int \sin 4x \cdot (1 - \cos 4x) dx &= \frac{1}{4} \left[ \int \sin 4x dx - \frac{1}{2} \int \sin 8x dx \right] \\ &= \frac{1}{4} \left[ -\frac{\cos 4x}{4} + \frac{\cos 8x}{16} \right]_{\pi/12}^{\pi/4} = \frac{1}{4} \left[ \frac{15}{32} \right] = \frac{15}{128} \end{aligned}$$

---

## Question249

**If**

$$\lim_{n \rightarrow \infty} \frac{1^a + 2^a + \dots + n^a}{(n+1)^{a-1} [(na+2) + \dots + (na+n)]} = \frac{1}{60}$$

**for some positive real number a, then a is equal to :**

**[Online April 9, 2017]**

### Options:

A. 7

B. 8

C.  $\frac{15}{2}$

D.  $\frac{17}{2}$

**Answer: A**

### Solution:

**Solution:**

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{(a+1)} \cdot n^{a+1} + a_1 n^a + a_2 n^{a-1} + \dots}{(n+1)^{a-1} \cdot n^2 \left( a + \frac{1 + \frac{1}{n}}{2} \right)} = \frac{1}{60}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \dots + \left(\frac{n}{n}\right)^a}{(n+1)^{a-1} \left[ n^2 a + \frac{n(n+1)}{2} \right]} = \frac{1}{60}$$

$$= \frac{\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n}\right)^a}{\left(1 + \frac{1}{n}\right)^{a-1} \left[ a + \frac{1}{2} \left(1 + \frac{1}{n}\right) \right]} = \frac{1}{60}$$

$$= \frac{\int_0^1 x^a dx}{\left(a + \frac{1}{2}\right)} = \frac{1}{60} = \frac{1}{a+1} = \frac{1}{a+12} = \frac{1}{60}$$

$$\Rightarrow \frac{1}{a+1} = \frac{1}{60}$$

$$\Rightarrow (a+1)(2a+1) = 120$$

$$\Rightarrow 2a^2 + 3a - 119 = 0$$

$$\Rightarrow 2a^2 + 17a - 14a - 119 = 0$$

$$\Rightarrow (a-7)(2a+17) = 0$$

$$\Rightarrow a = 7, -\frac{17}{2}$$

## Question250

If  $\int \frac{dx}{\cos^3 x \sqrt{2 \sin 2x}} = (\tan x)^A + C(\tan x)^B + k$ , where  $k$  is a constant of integration, then  $A + B + C$  equals:  
[Online April 9, 2016]

Options:

A.  $\frac{16}{5}$

B.  $\frac{27}{10}$

C.  $\frac{7}{10}$

D.  $\frac{21}{5}$

Answer: A

Solution:

Solution:

$$\int \frac{dx}{\cos^3 x \sqrt{4 \sin x \cos x}} = \int \frac{dx}{2 \cos^4 x \sqrt{\tan x}}$$

$$\text{Let } \tan x = t^2 \Rightarrow \sec^2 x = 1 + t^4$$

$$\sec^2 x dx = 2t dt$$

$$= \int \frac{\sec^4 x dx}{2\sqrt{\tan x}} = \int \frac{\sec^2 x (\sec^2 x dx)}{2\sqrt{\tan x}}$$

$$= \int \frac{(1+t^4)2t dt}{2t} = \int (1+t^4) dt = t + \frac{t^5}{5} + k$$

$$= \sqrt{\tan x} + \frac{1}{5} \tan^{5/2} x + k \quad [t = \sqrt{\tan x}]$$

$$A = \frac{1}{2}, B = \frac{5}{2}, C = \frac{1}{5}$$

$$A + B + C = \frac{16}{5}$$

---

## Question251

The integral  $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$  is equal to:

(where C is a constant of integration)

[Online April 10, 2016]

Options:

A.  $-2 \sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}} + C$

B.  $-\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} + C$

C.  $-2 \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} + C$

D.  $2 \sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}} + C$

Answer: C

Solution:

Solution:

$$I = \int \frac{dx}{(1+\sqrt{x}) \cdot \sqrt{x}\sqrt{1-x}}$$

$$\text{Put } 1+\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow I = \int \frac{2dt}{t\sqrt{2t-t^2}}$$

$$\text{Again put } t = \frac{1}{z} \Rightarrow dt = \frac{-1}{z^2} dz$$

$$\Rightarrow I = 2 \int \frac{\frac{-1}{z^2} dz}{\frac{1}{z} \sqrt{\frac{2}{z} - \frac{1}{z^2}}} = 2 \int \frac{-dz}{\sqrt{2z-1}}$$

$$= -2\sqrt{2z-1} + c = -2 \sqrt{\frac{2}{t} - 1} + c$$

$$= -2 \sqrt{\frac{2-t}{t}} + c = -2 \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} + c$$



## Question252

The integral  $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$  is equal to:

[2016]

Options:

A.  $\frac{x^5}{2(x^5 + x^3 + 1)^2} + C$

B.  $\frac{-x^{10}}{2(x^5 + x^3 + 1)^2} + C$

C.  $\frac{-x^5}{(x^5 + x^3 + 1)^2} + C$

D.  $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$

**Answer: D**

**Solution:**

**Solution:**

$$\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$$

Dividing by  $x^{15}$  in numerator and denominator

$$\int \frac{\frac{2}{x^3} + \frac{5}{x^6} dx}{\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^3}$$

$$\text{Let } 1 + \frac{1}{x^2} + \frac{1}{x^5} = t$$

$$\Rightarrow \left(\frac{-2}{x^3} - \frac{5}{x^6}\right) dx = dt \Rightarrow \left(\frac{2}{x^3} + \frac{5}{x^6}\right) dx = -dt$$

This gives,

$$\int \frac{\frac{2}{x^3} + \frac{5}{x^6} dx}{\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^3} = \int \frac{-dt}{t^3} = \frac{1}{2t^2} + C$$

$$= \frac{1}{2\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^2} + C = \frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$$


---

## Question 253

For  $x \in \mathbb{R}$ ,  $x \neq 0$ , if  $y(x)$  is a differentiable function such that

$$x \int_1^x y(t) dt = (x+1) \int_1^x ty(t) dt, \text{ then } y(x) \text{ equals:}$$

(where  $C$  is a constant)

[Online April 10, 2016]

Options:

A.  $Cx^3 e^{\frac{1}{x}}$

B.  $\frac{C}{x^2} e^{-\frac{1}{x}}$

C.  $\frac{C}{x} e^{-\frac{1}{x}}$

D.  $\frac{C}{x^3} e^{-\frac{1}{x}}$

**Answer: D**

**Solution:**

$$x \int_1^x y(t) dt = x \int_1^x ty(t) dt + \int_1^x ty(t) dt$$

Differentiate w.r to  $x$ .

$$\int_1^x y(t) dt + x[y(x) - y(1)]$$



$$= \int_1^x ty(t) dt + x[xy(x) - y(1)] + xy(x) - y(1)$$

$$\int_1^x y(t) dt = \int_1^x ty(t) dt + x^2y(x) - y(1)$$

Diff. again w. r. to x

$$y(x) - y(a) = xy(x) - y(a) + 2xy(x) + x^2y'(x)$$

$$(1 - 3x)y(x) = x^2y'(x)$$

$$\frac{y'(x)}{y(x)} = \frac{1 - 3x}{x^2}$$

$$\frac{1 dy}{y dx} = \frac{1 - 3x}{x^2} \Rightarrow \ln y = -\frac{1}{x} - 3 \ln x$$

$$\ln(yx^3) = -\frac{1}{x}$$

$$yx^3 = -e^{-1/x}$$

$$y = \frac{e^{-1/x}}{x^3} \text{ or } y = \frac{ce^{-1/x}}{x^3}$$

## Question254

The value of the integral  $\int_4^{10} \frac{[x^2]dx}{[x^2 - 28x + 196] + [x^2]}$ , where  $[x]$  denotes the greatest integer less than or equal to  $x$ , is :  
[Online April 10, 2016]

Options:

A.  $\frac{1}{3}$

B. 6

C. 7

D. 3

**Answer: D**

**Solution:**

$$I = \int_4^{10} \frac{[x^2]dx}{[x^2 - 28x + 196] + [x^2]} dx \dots\dots\dots(a)$$

Use  $\int_a^b f(a + b - x)dx = \int_a^b f(x)dx$

$$I = \int_4^{10} \frac{[(x - 14)^2]}{4[x^2] + [(x - 14)^2]} dx \dots\dots\dots(b)$$

(a) + (b)

$$2I = \int_4^{10} \frac{[(x - 14)^2] + [x^2]}{4[x^2] + [(x - 14)^2]} dx$$

$$2I = \int_4^{10} dx \Rightarrow 2I = 6 \Rightarrow I = 3$$

## Question255

If  $2 \int_0^1 \tan^{-1} x dx = \int_0^1 \cot^{-1}(1 - x + x^2) dx$ , then  $\int_0^1 \tan^{-1}(1 - x + x^2) dx$  is equal to:

[Online April 9, 2016]

Options:

A.  $\frac{\pi}{2} + \log 2$

B.  $\log 2$

C.  $\frac{\pi}{2} - \log 4$

D.  $\log 4$

**Answer: B**

**Solution:**

**Solution:**

$$2 \int_0^1 \tan^{-1} x dx = \int_0^1 \left( \frac{\pi}{2} - \tan^{-1}(1 - x + x^2) \right) dx$$

$$2 \int_0^1 \tan^{-1} x dx = \int_0^1 \frac{\pi}{2} dx - \int_0^1 \tan^{-1}(1 - x + x^2) dx$$

$$\int_0^1 \tan^{-1}(1 - x + x^2) dx = \frac{\pi}{2} - 2 \int_0^1 \tan^{-1} x dx \dots\dots\dots(a)$$

Let,  $I_1 = \int_0^1 \tan^{-1} x dx$

$$= [(\tan^{-1}x)x]_0^1 - \int_0^1 \frac{1}{1+x^2} x dx$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \frac{1}{2} \log 2$$

By equation (a)

$$\frac{\pi}{2} - 2 \left[ \frac{\pi}{4} - \frac{1}{2} \log 2 \right] = \log 2$$

## Question 256

$\lim_{n \rightarrow \infty} \left( \frac{(n+1)(n+2)\dots 3n}{n^{2n}} \right)^{\frac{1}{n}}$  is equal to:

[2016]

Options:

A.  $\frac{9}{e^2}$

B.  $3 \log 3 - 2$

C.  $\frac{18}{e^4}$

D.  $\frac{27}{e^2}$

**Answer: D**

**Solution:**

**Solution:**

$$y = \lim_{n \rightarrow \infty} \left( \frac{(n+1)(n+2)\dots 3n}{n^{2n}} \right)^{\frac{1}{n}}$$

$$\ln y = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( 1 + \frac{1}{n} \right) \left( 1 + \frac{2}{n} \right) \dots \left( 1 + \frac{2n}{n} \right)$$

$$\ln y = \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \ln \left( 1 + \frac{1}{n} \right) + \ln \left( 1 + \frac{2}{n} \right) + \dots + \ln \left( 1 + \frac{2n}{n} \right) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \ln \left( 1 + \frac{r}{n} \right) = \int_0^2 \ln(1+x) dx$$

Let  $1+x = t \Rightarrow dx = dt$

when  $x = 0, t = 1$

$x = 2, t = 3$

$$\ln y = \int_1^3 \ln t \, dt = [t \ln t - t]_1^3 = \ln(3^3 e^2) = \ln\left(\frac{27}{e^2}\right)$$

$$\Rightarrow y = \frac{27}{e^2}$$

---

## Question 257

If  $\int \frac{\log(t + \sqrt{1+t^2})}{\sqrt{1+t^2}} dt = \frac{1}{2}(g(t))^2 + C$ , where  $C$  is a constant, then  $g(b)$  is equal to :  
[Online April 11, 2015]

Options:

A.  $\frac{1}{\sqrt{5}} \log(2 + \sqrt{5})$

B.  $\frac{1}{2} \log(2 + \sqrt{5})$

C.  $2 \log(2 + \sqrt{5})$

D.  $\log(2 + \sqrt{5})$

**Answer: D**

**Solution:**

**Solution:**

$$\text{Let } I = \int \frac{\log(t + \sqrt{1+t^2})}{\sqrt{1+t^2}} dt$$

$$\text{put } u = \log(t + \sqrt{1+t^2})$$

$$du = \frac{1}{t + \sqrt{1+t^2}} \cdot \left[ \frac{t + \sqrt{1+t^2}}{\sqrt{1+t^2}} \right] dt = \frac{1}{\sqrt{1+t^2}} dt$$

$$\therefore I = \int u \, du = \frac{u^2}{2} + c$$

$$\text{Since, } I = \frac{1}{2}[g(t)]^2 + c$$



$$\therefore g(t) = \log(t + \sqrt{1+t^2})$$

Put  $t = 2$

$$g(b) = \log(2 + \sqrt{5})$$

---

## Question258

The integral  $\int \frac{dx}{x^2(x^4+1)^{3/4}}$  equals:

[2015]

Options:

A.  $-(x^4 + 1)^{\frac{1}{4}} + c$

B.  $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$

C.  $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$

D.  $(x^4 + 1)^{\frac{1}{4}} + c$

**Answer: B**

**Solution:**

**Solution:**

$$\begin{aligned} I &= \int \frac{dx}{x^2[x^4+1]^{\frac{3}{4}}} \\ &= \int \frac{dx}{x^2 \left[ (x^4)^{\frac{3}{4}} \left( 1 + \frac{1}{x^4} \right)^{\frac{3}{4}} \right]} \\ &= \int \frac{dx}{x^5 \left[ 1 + \frac{1}{x^4} \right]^{\frac{3}{4}}} \end{aligned}$$

Substitute:  $1 + \frac{1}{x^4} = t$

Differentiating w.r.t. x

$$0 - 4 \frac{1}{x^5} dx = dt$$

$$\Rightarrow \frac{dx}{x^5} = - \frac{dt}{4}$$

$$\begin{aligned} I &= \int \frac{\left(-\frac{dt}{4}\right)}{t^{\frac{3}{4}}} \\ &= -\frac{1}{4} \int t^{-\frac{3}{4}} dt \\ &= -\frac{1}{4} \frac{t^{-\frac{3}{4}+1}}{\left(-\frac{3}{4}+1\right)} + C \\ &= -\frac{1}{4} t^{\frac{1}{4}} + C \\ &= -\left[1 + \frac{1}{x^4}\right] \frac{1}{4} + C \\ I &= -\left[\frac{x^4+1}{x^4}\right] \frac{1}{4} + C \end{aligned}$$

---

## Question259

The integral  $\int \frac{dx}{(x+1)^{\frac{3}{4}}(x-2)^{\frac{5}{4}}}$  is equal to:

[Online April 10, 2015]

Options:

A.  $-\frac{4}{3} \left(\frac{x+1}{x-2}\right)^{\frac{1}{4}} + C$

B.  $4 \left(\frac{x+1}{x-2}\right)^{\frac{1}{4}} + C$

C.  $4 \left(\frac{x-2}{x+1}\right)^{\frac{1}{4}} + C$



$$D. -\frac{4}{3} \left( \frac{x-2}{x+1} \right)^{\frac{1}{4}} + C$$

**Answer: B**

**Solution:**

**Solution:**

$$\int \frac{dx}{(x+1)^{3/4}(x-2)^{5/4}}$$

$$\int \frac{dx}{\left(\frac{x+1}{x-2}\right)^{3/4}(x-2)^2}, \text{ put } \frac{x+1}{x-2} = t$$

$$\frac{-3}{(x-2)^2} = \frac{dt}{dx}$$

$$\frac{dx}{(x-2)^2} = -\frac{dt}{3} = \frac{-1}{3} \int \frac{dt}{t^4} = -\frac{1}{3} \int t^{-3} dt$$

$$= \frac{1}{3} \left[ \frac{-3}{-4} t^{-2} + 1 \right] = \frac{-4}{3} \left[ \frac{x+1}{x-2} \right]^{-2} + c$$

## Question 260

The integral  $\int_2^4 \frac{\log x^2}{2 \log x^2 + \log(36 - 12x + x^2)} dx$  is equal to  
[2015]

**Options:**

- A. 1
- B. 6
- C. 2
- D. 4

**Answer: A**

**Solution:**



$$I = \int_2^4 \frac{\log x^2}{2 \log x^2 + \log(36 - 12x + x^2)} dx$$

$$I = \int_2^4 \frac{\log x^2}{2 \log x^2 + \log(6-x)^2} dx \dots\dots(i)$$

$$I = \int_2^4 \frac{\log(6-x)^2}{2 \log(6-x)^2 + \log x^2} dx \dots\dots(ii)$$

Adding (i) and (ii)

$$2I = \int_2^4 dx = [x]_2^4 = 2$$

$$I = 1$$

## Question261

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that  $f(2-x) = f(2+x)$  and  $f(4-x) = f(4+x)$ , for all  $x \in \mathbb{R}$  and  $\int_0^2 f(x) dx = 5$ . Then the value of

$\int_{10}^{50} f(x) dx$  is :

[Online April 11, 2015]

Options:

A. 125

B. 80

C. 100

D. 200

**Answer: C**

**Solution:**

**Solution:**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that  $f(2-x) = f(2+x)$

Put  $x = 2+x$  we get

$$f(-x) = f(4+x) = f(4-x)$$

$$\Rightarrow f(x) = f(x+4)$$

Hence period is 4

$$\text{Consider } \int_{10}^{50} f(x) dx = 10 \int_{10}^{14} f(x) dx = 10[5 + 5] = 100$$

## Question262

Let  $f : (-1, 1) \rightarrow \mathbf{R}$  be a continuous function. If  $\int_0^x f(t) dt = \frac{\sqrt{3}}{2}x$ , then

$f\left(\frac{\sqrt{3}}{2}\right)$  is equal to :

[Online April 11, 2015]

Options:

A.  $\frac{1}{2}$

B.  $\frac{\sqrt{3}}{2}$

C.  $\sqrt{\frac{3}{2}}$

D.  $\sqrt{3}$

**Answer: D**

**Solution:**

**Solution:**

Let  $f : (-1, 1) \rightarrow \mathbf{R}$  be a continuous function

$$\text{Let } \int_0^{\sin x} f(t) dt = \frac{\sqrt{3}}{2}x$$

$$f(\sin x) \cdot \frac{d}{dx}(\sin x) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow f(\sin x) \cdot \cos x = \frac{\sqrt{3}}{2}$$

$$\text{put } x = \frac{\pi}{3}$$

$$f\left(\sin \frac{\pi}{3}\right) \cdot \cos \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{\sqrt{3}}{2}\right) \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

## Question263

For  $x > 0$ , let  $f(x) = \int_1^x \frac{\log t}{1+t} dt$ . Then  $f(x) + f\left(\frac{1}{x}\right)$  is equal to:

[Online April 10, 2015]

Options:

A.  $\frac{1}{4}(\log x)^2$

B.  $\log x$

C.  $\frac{1}{2}(\log x)^2$

D.  $\frac{1}{4} \log x^2$

Answer: C

Solution:

Solution:

$$f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ln t}{1+t} dt$$

$$\text{Let } t = \frac{1}{z}$$

$$dt = -\frac{1}{z^2} dz$$

$$f(x) = \int_1^x \frac{\ln z}{z(z+1)} dz$$

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln x}{z} dz = \left[ \frac{(\ln z)^2}{2} \right]_1^x = \frac{(\ln x)^2}{2}$$

---

## Question264

The integral  $\int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx$  is equal to

[2014]



**Options:**

A.  $(x + 1)e^{x + \frac{1}{x}} + c$

B.  $-xe^{x + \frac{1}{x}} + c$

C.  $(x - 1)e^{x + \frac{1}{x}} + c$

D.  $xe^{x + \frac{1}{x}} + c$

**Answer: D**

**Solution:**

**Solution:**

$$\begin{aligned}\text{Let } I &= \int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx \\ &= \int e^{x + \frac{1}{x}} dx + \int \left(x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx \\ &= x \cdot e^{x + \frac{1}{x}} - \int x \left(1 - \frac{1}{x^2}\right) e^{x + \frac{1}{x}} dx + \int \left(x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx \\ &= x \cdot e^{x + \frac{1}{x}} - \int \left(x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx + \int (x - 1x) e^{x + \frac{1}{x}} dx \\ &= x e^{x + \frac{1}{x}} + C\end{aligned}$$

---

## Question 265

The integral  $\int \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$  is equal to:

[Online April 12, 2014]

**Options:**

A.  $\frac{1}{(1 + \cot^3 x)} + c$

$$B. -\frac{1}{3(1 + \tan^3 x)} + c$$

$$C. \frac{\sin^3 x}{(1 + \cos^3 x)} + c$$

$$D. -\frac{\cos^3 x}{3(1 + \sin^3 x)} + c$$

**Answer: B**

**Solution:**

**Solution:**

$$\text{Let } I = \int \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$$

$$I = \int \left( \frac{\sin x \cdot \cos x}{\sin^3 x + \cos^3 x} \right)^2 dx$$

$$I = \int \left( \frac{\sin x \cdot \cos x}{\cos^2 x (1 + \tan^3 x)} \right)^2 dx$$

$$= \int \left( \frac{\sin x \cdot \sec^2 x}{(1 + \tan^3 x)} \right)^2 dx$$

$$\text{Put } 1 + \tan^3 x = t$$

$$dt = 3 \tan^2 x \sec^2 x dx \text{ or } dx = \frac{dt}{3 \tan^2 x \sec^2 x}$$

$$\therefore I = \int \frac{\sin^2 x \cdot \sec^4 x}{t^2} \times \frac{dt}{3 \tan^2 x \sec^2 x}$$

$$I = \frac{1}{3} \int \frac{\sin^2 x \cdot \sec^4 x}{t^2} \times \frac{dt}{\frac{\sin^2 x}{\cos^2 x} \times \sec^2 x}$$

$$= \frac{1}{3} \int \frac{\sin^2 x \cdot \sec^4 x}{t^2} \times \frac{dt}{\sin^2 x \sec^4 x}$$

$$\therefore I = \frac{1}{3} \int \frac{dt}{t^2} = \frac{1}{3} \int t^{-2} dt$$

$$I = \frac{1}{3} \left[ \frac{t^{-2+1}}{-2+1} \right] + c = \frac{-1}{3} \left[ \frac{1}{t} \right] + c$$

$$\text{or } I = -\frac{1}{3(1 + \tan^3 x)} + c$$

## Question266

The integral  $\int x \cos^{-1} \left( \frac{1-x^2}{1+x^2} \right) dx$  ( $x > 0$ ) is equal to:

[Online April 11, 2014]

Options:

A.  $-x + (1+x^2)\tan^{-1}x + c$

B.  $x - (1+x^2)\cot^{-1}x + c$

C.  $-x + (1+x^2)\cot^{-1}x + c$

D.  $x - (1+x^2)\tan^{-1}x + c$

Answer: A

Solution:

Solution:

$$\text{Let } I = \int x \cos^{-1} \left( \frac{1-x^2}{1+x^2} \right) dx$$

$$\therefore I = 2 \int x \cdot \tan^{-1}x dx$$

Applying Integration by parts

$$I = 2 \left[ \tan^{-1}x \int x dx - \int \left( \frac{d}{dx}(\tan^{-1}x) \int x dx \right) dx \right]$$

$$I = 2 \left[ \frac{x^2}{2} \tan^{-1}x - \int \frac{1}{1+x^2} \times \frac{x^2}{2} dx \right] + c$$

$$I = 2 \left[ \frac{x^2}{2} \tan^{-1}x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} dx \right] + c$$

$$I = 2 \left[ \frac{x^2}{2} \tan^{-1}x - \frac{1}{2} \int \frac{x^2+1}{x^2+1} dx + \frac{1}{2} \int \frac{1}{1+x^2} dx \right] + c$$

$$I = 2 \left[ \frac{x^2}{2} \tan^{-1}x - \frac{1}{2} \int 1 \cdot dx + \frac{1}{2} \tan^{-1}x \right] + c$$

$$I = 2 \left[ \frac{x^2}{2} \tan^{-1}x - \frac{x}{2} + \frac{1}{2} \tan^{-1}x \right] + c$$

$$I = x^2 \tan^{-1}x + \tan^{-1}x - x + c$$

$$\text{or } I = -x + (x^2+1)\tan^{-1}x + c$$

---

Question267

$\int \frac{\sin^8 x - \cos^8 x}{(1 - 2\sin^2 x \cos^2 x)} dx$  is equal to:

[Online April 9, 2014]

Options:

A.  $\frac{1}{2} \sin 2x + c$

B.  $-\frac{1}{2} \sin 2x + c$

C.  $-\frac{1}{2} \sin x + c$

D.  $-\sin^2 x + c$

Answer: B

Solution:

Solution:

$$\begin{aligned} \text{Let } I &= \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx \\ &= \int \frac{(\sin^4 x)^2 - (\cos^4 x)^2}{1 - 2\sin^2 x \cos^2 x} dx \\ &= \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{1 - 2\sin^2 x \cos^2 x} dx \\ &= \int \frac{[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x][(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)]}{1 - 2\sin^2 x \cos^2 x} dx \\ &= -\int \cos 2x dx = \frac{-\sin 2x}{2} + c = -\frac{1}{2} \sin 2x + c \end{aligned}$$

## Question 268

If  $m$  is a non-zero number and  $\int \frac{x^{5m-1} + 2x^{4m-1}}{(x^{2m} + x^m + 1)^3} dx = f(x) + c$ , then  $f(x)$  is:

[Online April 19, 2014]

Options:

A.  $\frac{x^{5m}}{2m(x^{2m} + x^m + 1)^2}$



$$B. \frac{x^{4m}}{2m(x^{2m} + x^m + 1)^2}$$

$$C. \frac{2m(x^{5m} + x^{4m})}{(x^{2m} + x^m + 1)^2}$$

$$D. \frac{(x^{5m} - x^{4m})}{2m(x^{2m} + x^m + 1)^2}$$

**Answer: B**

**Solution:**

**Solution:**

$$\int \frac{x^{5m-1} + 2x^{4m-1}}{(x^{2m} + x^m + 1)^3} dx = \int \frac{x^{5m-1} + 2x^{4m-1}}{x^{6m}(1 + x^{-m} + x^{-2m})^3} dx$$

$$= \int \frac{x^{-m-1} + 2x^{-2m-1}}{(1 + x^{-m} + x^{-2m})^3} dx$$

$$\text{Put } t = 1 + x^{-m} + x^{-2m}$$

$$\therefore \frac{dt}{dx} = -mx^{-m-1} - 2mx^{-2m-1}$$

$$\Rightarrow \frac{dt}{-m} = (x^{-m-1} + 2x^{-2m-1}) dx$$

$$\therefore \int \frac{x^{5m-1} + 2x^{4m-1}}{(x^{2m} + x^m + 1)^3} dx = \frac{1}{-m} \int t^{-3} dt = \frac{1}{2mt^2} + C$$

$$= \frac{1}{2m(1 + x^{-m} + x^{-2m})^2} + C$$

$$= \frac{x^{4m}}{2m(x^{2m} + x^m + 1)^2} + C$$

$$\therefore f(x) = \frac{x^{4m}}{2m(x^{2m} + x^m + 1)^2}$$

## Question 269

The integral  $\int_0^{\pi} \sqrt{1 + 4\sin^2 \frac{x}{2} - 4\sin \frac{x}{2}} dx$  equals:

[2014]

Options:

A.  $4\sqrt{3} - 4$

B.  $4\sqrt{3} - 4 - \frac{\pi}{3}$

C.  $\pi - 4$

D.  $\frac{2\pi}{3} - 4 - 4\sqrt{3}$

**Answer: B**

**Solution:**

**Solution:**

$$\begin{aligned} \text{Let } I &= \int_0^{\pi} \sqrt{1 + 4\sin^2 \frac{x}{2} - 4\sin \frac{x}{2}} dx = \int_0^{\pi} \left| 2\sin \frac{x}{2} - 1 \right| dx \\ &= \int_0^{\pi/3} \left( 1 - 2\sin \frac{x}{2} \right) dx + \int_{\pi/3}^{\pi} \left( 2\sin \frac{x}{2} - 1 \right) dx \\ &\left[ \because \sin \frac{x}{2} = \frac{1}{2} \Rightarrow \frac{x}{2} = \frac{\pi}{6} \Rightarrow x = \frac{\pi}{3}, \frac{x}{2} = \frac{5\pi}{6} \Rightarrow x = \frac{5\pi}{3} > \pi \right] \\ &= \left[ x + 4\cos \frac{x}{2} \right]_0^{\pi/3} + \left[ -4\cos \frac{x}{2} - x \right]_{\pi/3}^{\pi} \\ &= \frac{\pi}{3} + 4\frac{\sqrt{3}}{2} - 4 + \left( 0 - \pi + 4\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) \\ &= 4\sqrt{3} - 4 - \frac{\pi}{3} \end{aligned}$$

---

## Question270

Let function  $F$  be defined as  $F(x) = \int_1^x \frac{e^t}{t} dt$ ,  $x > 0$  then the value of the integral  $\int_1^x \frac{e^t}{t+a} dt$ , where  $a > 0$ , is:

**[Online April 19, 2014]**

**Options:**

A.  $e^a[F(x) - F(1+a)]$

B.  $e^{-a}[F(x+a) - F(a)]$

C.  $e^a[F(x+a) - F(1+a)]$

$$D. e^{-a}[F(x+a) - F(1+a)]$$

**Answer: D**

**Solution:**

**Solution:**

$$F(x) = \int_1^x \frac{e^t}{t} dt, x > 0$$

$$\text{Let } I = \int_1^x \frac{e^t}{t+a} dt$$

$$\text{Put } t+a = z \Rightarrow t = z-a; dt = dz$$

$$\text{for } t = 1, z = 1+a$$

$$\text{for } t = x, z = x+a$$

$$\therefore I = \int_{1+a}^{x+a} \frac{e^{z-a}}{z} dz$$

$$= e^{-a} \int_{1+a}^{x+a} \frac{e^z}{z} dz \equiv e^{-a} \int_{1+a}^{x+a} \frac{e^t}{t} dt$$

$$I = e^{-a} \left[ \int_{1+a}^1 \frac{e^t}{t} dt + \int_1^{x+a} \frac{e^t}{t} dt \right]$$

$$= e^{-a} \left[ - \int_1^{1+a} \frac{e^t}{t} dt + \int_1^{x+a} \frac{e^t}{t} dt \right]$$

$$= e^{-a} [-F(1+a) + F(x+a)]$$

(By the definition of  $F(x)$ )

$$= e^{-a} [F(x+a) - F(1+a)]$$

---

## Question271

If for a continuous function  $f(x)$ ,  $\int_{-\pi}^t (f(x) + x) dx = \pi^2 - t^2$ , for all  $t \geq -\pi$ ,

then  $f\left(-\frac{\pi}{3}\right)$  is equal to:

[Online April 12, 2014]

**Options:**

A.  $\pi$

B.  $\frac{\pi}{2}$

C.  $\frac{\pi}{3}$

D.  $\frac{\pi}{6}$

**Answer: A**

**Solution:**

**Solution:**

$$\text{Let } \int_{-\pi}^t (f(x) + x) dx = \pi^2 - t^2$$

$$\Rightarrow \int_{-\pi}^t f(x) dx + \int_{-\pi}^t x dx = \pi^2 - t^2$$

$$\Rightarrow \int_{-\pi}^t f(x) dx + \left( \frac{t^2}{2} - \frac{\pi^2}{2} \right) = \pi^2 - t^2$$

$$\Rightarrow \int_{-\pi}^t f(x) dx = \frac{3}{2}(\pi^2 - t^2)$$

differentiating with respect to t

$$\frac{d}{dt} \left[ \int_{-\pi}^t f(x) dx \right] = \frac{3}{2} \frac{d}{dt} (\pi^2 - t^2)$$

$$f(t) \cdot \frac{dt}{dt} - f(-\pi) \frac{d}{dt} (-\pi) = -3t$$

$$f(t) = -3t$$

$$f\left(-\frac{\pi}{3}\right) = -3\left(-\frac{\pi}{3}\right) = \pi$$

---

## Question272

If  $[ ]$  denotes the greatest integer function, then the integral  $\int_0^{\pi} [\cos x] dx$  is equal to:

[Online April 12, 2014]

**Options:**

A.  $\frac{\pi}{2}$

B. 0

C. -1

D.  $-\frac{\pi}{2}$

**Answer: D**

**Solution:**

Let  $I = \int_0^{\pi} [\cos x] dx \dots\dots(1)$

$I = \int_0^{\pi} [\cos(\pi - x)] dx = \int_0^{\pi} [-\cos x] dx \dots\dots(2)$

On adding (1) and (2), we get

$2I = \int_0^{\pi} [\cos x] dx + \int_0^{\pi} [-\cos x] dx$

$2I = \int_0^{\pi} [\cos x] + [-\cos x] dx$

$2I = \int_0^{\pi} -1 dx \quad (\because [x] + [-x] = -1 \text{ if } x \notin Z)$

$2I = -x \Big|_0^{\pi} = -\pi$

$\Rightarrow I = \frac{-\pi}{2}$

---

## Question273

If for  $n \geq 1$ ,  $P_n = \int_1^e (\log x)^n dx$ , then  $P_{10} - 90P_8$  is equal to:  
[Online April 11, 2014]

**Options:**

A. -9

B. 10e

C. -9e

D. 10

**Answer: C**

**Solution:**

$$P_n = \int_1^e (\log x)^n dx$$

put  $\log x = t$  then  $x = e^t$  and  $dx = e^t dt$

Also, when  $x = 1$ , then  $t = \log 1 = 0$

and when  $x = e$ , then  $t = \log_e e = 1$

$$\therefore P_n = \int_0^1 t^n \cdot e^t dt$$

$$\therefore P_{10} = \int_0^1 t^{10} e^t dt \text{ and } P_8 = \int_0^1 t^8 e^t dt$$

$$\text{Now, } P_{10} - 90P_8 = \int_0^1 t^{10} e^t dt - 90 \int_0^1 t^8 e^t dt$$

$$P_{10} - 90P_8 = [t^{10} e^t]_0^1 - 10 \int_0^1 t^9 e^t dt - 90 \int_0^1 t^8 e^t dt$$

$$P_{10} - 90P_8 = e - 10 \left[ t^9 \int_0^1 e^t dt - \int_0^1 \frac{d}{dt}(t^9) \int_0^1 e^t dt \right] - 90 \int_0^1 t^8 e^t dt$$

$$P_{10} - 90P_8 = e - 10 \left[ e - 9 \int_0^1 t^8 e^t dt \right] - 90 \int_0^1 t^8 e^t dt$$

$$P_{10} - 90P_8 = e - 10e + 90 \int_0^1 t^8 e^t dt - 90 \int_0^1 t^8 e^t dt$$

$$\therefore P_{10} - 90P_8 = -9e$$

## Question 274

The integral  $\int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx$ , equals:

[Online April 9, 2014]

Options:

A.  $\frac{\pi}{4} \ln 2$

B.  $\frac{\pi}{8} \ln 2$

C.  $\frac{\pi}{16} \ln 2$

D.  $\frac{\pi}{32} \ln 2$

**Answer: C**

## Solution:

**Solution:**

$$\text{Let } I = \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx \text{ or } = \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+(2x)^2} dx$$

Put  $2x = \tan \theta$

$$\therefore \frac{2dx}{d\theta} = \sec^2 \theta \text{ or } dx = \frac{\sec^2 \theta d\theta}{2}$$

also when  $x = 0 \Rightarrow \theta = 0$

and when  $x = \frac{1}{2} \Rightarrow \theta = 45^\circ \text{ or } \frac{\pi}{4}$

$$\therefore I = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{1+\tan^2 \theta} \times \frac{\sec^2 \theta d\theta}{2}$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{1+\tan^2 \theta} \times \sec^2 \theta d\theta \text{ (because } 1+\tan^2 \theta = \sec^2 \theta)$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(1+\tan \theta) d\theta \dots\dots(i)$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left[ 1 + \tan \left( \frac{\pi}{4} - \theta \right) \right] d\theta \text{ (Using the property of definite integral)}$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left[ 1 + \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \times \tan \theta} \right] d\theta$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left[ 1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right] d\theta$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left[ \frac{1 + \tan \theta + 1 - \tan \theta}{1 + \tan \theta} \right] d\theta$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left[ \frac{2}{1 + \tan \theta} \right] d\theta$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} [\ln 2 - \ln(1 + \tan \theta)] d\theta$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln 2 \cdot d\theta - \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(1 + \tan \theta) d\theta$$

$$I = \frac{1}{2} \ln 2 \theta \Big|_0^{\pi/4} - I \text{ (from eq. (i))}$$

$$I + I = \frac{1}{2} \ln 2 \left( \frac{\pi}{4} - 0 \right)$$

$$2I = \frac{1}{2} \times \frac{\pi}{4} \times \ln 2$$

$$2I = \frac{\pi}{8} \ln 2 \text{ or } I = \frac{\pi}{16} \ln 2$$

---

## Question 275

If  $\int f(x) dx = \psi(x)$ , then  $\int x^5 f(x^3) dx$  is equal to  
[2013]

Options:

A.  $\frac{1}{3}[x^3 \psi(x^3) - \int x^2 \psi(x^3) dx] + C$

B.  $\frac{1}{3}x^3 \psi(x^3) - 3 \int x^3 \psi(x^3) dx + C$

C.  $\frac{1}{3}x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + C$

D.  $\frac{1}{3}[x^3 \psi(x^3) - \int x^3 \psi(x^3) dx] + C$

Answer: C

Solution:

Solution:

Let  $\int f(x) dx = \psi(x)$

Let  $I = \int x^5 f(x^3) dx$

put  $x^3 = t$

$$\Rightarrow 3x^2 dx = dt$$

$$I = \frac{1}{3} \int 3 \cdot x^2 \cdot x^3 \cdot f(x^3) \cdot dx$$

$$= \frac{1}{3} \int t f(t) dt = \frac{1}{3} [t \int f(t) dt - \int f(t) dt]$$

$$= \frac{1}{3} [t \psi(t) - \int \psi(t) dt]$$

$$= \frac{1}{3} [x^3 \psi(x^3) - 3 \int x^2 \psi(x^3) dx] + c$$

$$= \frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + c$$



## Question276

If the integral  $\int \frac{\cos 8x + 1}{\cot 2x - \tan 2x} dx = A \cos 8x + k$

where k is an arbitrary constant, then A is equal to :  
[Online April 25, 2013]

Options:

A.  $-\frac{1}{16}$

B.  $\frac{1}{16}$

C.  $\frac{1}{8}$

D.  $-\frac{1}{8}$

Answer: A

Solution:

Solution:

$$\text{Let } I = \int \frac{\cos 8x + 1}{\cot 2x - \tan 2x} dx$$

$$\text{Now, } D^r = \cot 2x - \tan 2x = \frac{\cos 2x}{\sin 2x} - \frac{\sin 2x}{\cos 2x}$$

$$= \frac{\cos^2 2x - \sin^2 2x}{\sin 2x \cos 2x} = \frac{2 \cos 4x}{\sin 4x}$$

$$\therefore I = \int \frac{2 \cos^2 4x}{\frac{2 \cos 4x}{\sin 4x}} dx = \int \frac{2 \cos^2 4x \cdot \sin 4x}{2 \cos 4x} dx$$

$$= \frac{1}{2} \int \sin 8x dx = -\frac{1}{2} \frac{\cos 8x}{8} + k = -\frac{1}{16} \cdot \cos 8x + k$$

$$\text{Now, } -\frac{1}{16} \cdot \cos 8x + k = A \cos 8x + k$$

$$\Rightarrow A = -\frac{1}{16}$$

---

## Question277

The integral  $\int \frac{x dx}{2-x^2 + \sqrt{2-x^2}}$  equals:

[Online April 23, 2013]

Options:

A.  $\log|1 + \sqrt{2+x^2}|+c$

B.  $-\log|1 + \sqrt{2-x^2}|+c$

C.  $x \log|1 - \sqrt{2+x^2}|+c$

D.  $-x \log|1 - \sqrt{2-x^2}|+c$

Answer: B

Solution:

Solution:

$$I = \int \frac{x dx}{2-x^2 + \sqrt{2-x^2}}$$

$$\text{Put } t = \sqrt{2-x^2}, \frac{dt}{dx} = \frac{1}{2\sqrt{2-x^2}} \cdot (-2x)$$

$$\Rightarrow -tdt = x dx$$

$$\therefore I = \int \frac{(-t)dt}{t^2+t} = -\int \frac{1}{t+1} dt = -\log|t+1|$$

$$= -\log|1 + \sqrt{2-x^2}|+c$$

---

## Question278

If  $\int \frac{x^2-x+1}{x^2+1} e^{\cot^{-1}x} dx = A(x)e^{\cot^{-1}x} + C$ , then A(x) is equal to :

[Online April 22, 2013]

Options:

A.  $-x$

B.  $x$

C.  $\sqrt{1-x}$

D.  $\sqrt{1+x}$

**Answer: B**

**Solution:**

**Solution:**

$$\text{Let } I = \int \frac{x^2 - x + 1}{x^2 + 1} \cdot e^{\cot^{-1}x} dx$$

$$\text{Put } x = \cot t \Rightarrow -\operatorname{cosec}^2 t dt = dx$$

$$\text{Now, } 1 + \cot^2 t = \operatorname{cosec}^2 t$$

$$\therefore I = \int \frac{e^t(\cot^2 t - \cot t + 1)}{(1 + \cot^2 t)} (-\operatorname{cosec}^2 t) dt$$

$$= -\int e^t(\operatorname{cosec}^2 t - \cot t) dt$$

$$= \int e^t(\cot t - \operatorname{cosec}^2 t) dt = e^t \cot t + C$$

$$= e^{\cot^{-1}x}(x) + C \equiv A(x) \cdot e^{\cot^{-1}x} + C$$

$$\Rightarrow A(x) = x$$

## Question279

If  $\int dx \frac{x^6}{x+x^7} = p(x)$  then  $\int \frac{x^6}{x+x^7} dx$  is equal to:

[Online April 9, 2013]

**Options:**

A.  $\ln |x| - p(x) + c$

B.  $\ln |x| + p(x) + c$

C.  $x - p(x) + c$

D.  $x + p(x) + c$

**Answer: A**

**Solution:**

$$\int \frac{x^6}{x+x^7} dx = \int \frac{x^6}{x(1+x^6)} dx = \int \frac{(1+x^6)-1}{x(1+x^6)} dx$$

$$= \int \frac{1}{x} dx - \int \frac{1}{1+x^6} dx = \ln|x| - p(x) + c$$


---

## Question280

The intercepts on x -axis made by tangents to the curve,  $y = \int_0^x |t| dt$ ,  $x \in \mathbb{R}$ , which are parallel to the line  $y = 2x$ , are equal to :

[2013]

Options:

- A.  $\pm 1$
- B.  $\pm 2$
- C.  $\pm 3$
- D.  $\pm 4$

Answer: A

Solution:

Solution:

Since,  $y = \int_0^x |t| dt$ ,  $x \in \mathbb{R}$

therefore  $\frac{dy}{dx} = |x|$

But from  $y = 2x$ ,  $\therefore \frac{dy}{dx} = 2$

$\Rightarrow |x| = 2 \Rightarrow x = \pm 2$

Points  $y = \int_0^{\pm 2} |t| dt = \pm 2$

$\therefore$  Equation of tangent is

$y - 2 = 2(x - 2)$  or  $y + 2 = 2(x + 2)$

$\Rightarrow$  x -intercept =  $\pm 1$

---

# Question281

**Statement-1 : The value of the integral**

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}} \text{ is equal to } \pi/6$$

**Statement-2 :  $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$**

**[2013]**

**Options:**

A. Statement-1 is true; Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

B. Statement-1 is true; Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

C. Statement-1 is true; Statement-2 is false.

D. Statement-1 is false; Statement-2 is true.

**Answer: D**

**Solution:**

**Solution:**

$$\begin{aligned} \text{Let, } I &= \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}} \\ &= \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x} dx}{1 + \sqrt{\tan x}} \dots\dots(i) \end{aligned}$$

Also, given

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x} dx}{1 + \sqrt{\tan x}} \dots\dots(ii)$$

By adding (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} dx$$

$$\Rightarrow I = \frac{1}{2} \left[ \frac{\pi}{3} - \frac{\pi}{6} \right] = \frac{\pi}{12}$$

Statement-1 is false

$$\therefore \int_a^b f(x)dx = \int_a^b f(a + b - x)dx$$

It is fundamental property.

Statement -2 is true.

## Question282

For  $0 \leq x \leq \frac{\pi}{2}$ , the value of  $\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t}) dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t}) dt$  equals :  
[Online April 25, 2013]

Options:

- A.  $\frac{\pi}{4}$
- B. 0
- C. 1
- D.  $-\frac{\pi}{4}$

Answer: A

Solution:

Solution:

Consider

$$\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t}) dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t}) dt$$

Let  $I = f(x)$  after integrating and putting the limits.

$$f'(x) = \sin^{-1} \sqrt{\sin^2 x} (2 \sin x \cos x) - 0 + \cos^{-1} \sqrt{\cos^2 x} (-2 \cos x \sin x) - 0$$

$$\therefore f'(x) = 0 \Rightarrow f(x) = C \text{ (constant)}$$

Now, we find  $f(x)$  at  $x = \frac{\pi}{4}$

$$\therefore I = \int_0^{1/2} \sin^{-1} \sqrt{t} dt + \int_0^{1/2} \cos^{-1} \sqrt{t} dt$$

$$= \int_0^{1/2} (\sin^{-1} \sqrt{t} + \cos^{-1} \sqrt{t}) dt = \int_0^{1/2} \frac{\pi}{2} dt = \frac{\pi}{4} = C$$

$$\therefore f(x) = \frac{\pi}{4}$$

$$\therefore \text{Required integration} = \frac{\pi}{4}$$

---

## Question283

The value of  $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$  is :

[Online April 23, 2013]

Options:

A.  $\pi$

B.  $\frac{\pi}{2}$

C.  $4\pi$

D.  $\frac{\pi}{4}$

Answer: D

Solution:

Solution:

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx \dots\dots(i)$$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}} dx, \text{ by replacing } x \text{ by } \left(\frac{\pi}{2} - \frac{\pi}{2} - x\right)$$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{2^x \cdot \sin^2 x}{1+2^x} dx \dots\dots(ii)$$

Adding equations (i) and (ii), we get

$$2I = \int_{-\pi/2}^{\pi/2} \sin^2 x dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 - \cos 2x) dx$$

$$\Rightarrow I = \frac{1}{4} \left[ x + \frac{\sin 2x}{2} \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{4} \left[ \left( \frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left( -\frac{\pi}{2} + \frac{\sin(-\pi)}{2} \right) \right]$$

$$\Rightarrow I = \frac{1}{4} \left[ \frac{\pi}{2} + \frac{\pi}{2} \right] = \frac{\pi}{4}$$

---

## Question284

The integral  $\int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x} dx$  is equal to :

[Online April 22, 2013]

**Options:**

A.  $\log 2 \sqrt{2}$

B.  $\log 2$

C.  $2 \log 2$

D.  $\log \sqrt{2}$

**Answer: D**

**Solution:**

**Solution:**

$$\begin{aligned}\text{Let } I &= \int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x} dx \\ &= \int_{7\pi/4}^{7\pi/3} \tan x dx = -\log \cos x \Big|_{7\pi/4}^{7\pi/3} \\ &= -\left[ \log \cos \frac{7\pi}{3} - \log \cos \frac{7\pi}{4} \right] \\ &= \log \cos \frac{7\pi}{4} - \log \cos \frac{7\pi}{3} \\ &= \log \left[ \frac{\cos \frac{7\pi}{4}}{\cos \frac{7\pi}{3}} \right] = \log \left[ \frac{\cos \left( 2\pi - \frac{\pi}{4} \right)}{\cos \left( 2\pi + \frac{\pi}{3} \right)} \right] \\ &= \log \left( \frac{\cos \frac{\pi}{4}}{\cos \frac{\pi}{3}} \right) = \log \left( \frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}} \right) \\ &= \log \left( \frac{2}{\sqrt{2}} \right) = \log \sqrt{2}\end{aligned}$$

---

## Question285

If  $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$ , then  $\frac{d^2y}{dx^2}$  is equal to:

[Online April 9, 2013]

**Options:**



A.  $y$

B.  $\sqrt{1+y^2}$

C.  $\frac{x}{\sqrt{1+y^2}}$

D.  $y^2$

**Answer: A**

**Solution:**

**Solution:**

$$x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$$

$$\Rightarrow 1 = \frac{1}{\sqrt{1+y^2}} \cdot \frac{dy}{dx}$$

$$\left[ \because \text{If } I(x) = \int_{\phi(x)}^{\Psi(x)} f(t) dt, \text{ then } \frac{dI(x)}{dx} = f\{\Psi(x)\} \left\{ \frac{d}{dx} \Psi(x) \right\} - f\{\phi(x)\} \cdot \left\{ \frac{d}{dx} \phi(x) \right\} \right]$$

$$\frac{dy}{dx} = \sqrt{1-y^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{1+y^2}} \cdot 2y \cdot \frac{dy}{dx} = \frac{y}{\sqrt{1+y^2}} \cdot \sqrt{1+y^2} = y$$

---

## Question286

If the  $\int \frac{5 \tan x}{\tan x - 2} dx = x + a \ln |\sin x - 2 \cos x| + k$ , then  $a$  is equal to :

[2012]

**Options:**

A. -1

B. -2

C. 1

D. 2

**Answer: D**

**Solution:**

**Solution:**

$$\begin{aligned}\int \frac{5 \tan x}{\tan x - 2} dx &= \int \frac{5 \frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} - 2} dx \\ &= \int \left( \frac{5 \sin x}{\cos x} \times \frac{\cos x}{\sin x - 2 \cos x} \right) dx \\ &= \int \frac{5 \sin x dx}{\sin x - 2 \cos x} \\ &= \int \left( \frac{4 \sin x + \sin x + 2 \cos x - 2 \cos x}{\sin x - 2 \cos x} \right) dx \\ &= \int \frac{(\sin x - 2 \cos x) + (4 \sin x + 2 \cos x)}{\sin x - 2 \cos x} dx \\ &= \int \frac{(\sin x - 2 \cos x) + 2(\cos x + 2 \sin x)}{(\sin x - 2 \cos x)} dx \\ &= \int \frac{\sin x - 2 \cos x}{\sin x - 2 \cos x} dx + 2 \int \left( \frac{\cos x + 2 \sin x}{\sin x - 2 \cos x} \right) dx \\ &= \int dx + 2 \int \frac{\cos x + 2 \sin x}{\sin x - 2 \cos x} dx = I_1 + I_2\end{aligned}$$

where,  $I_1 = \int dx$  and  $I_2 = 2 \int \frac{\cos x + 2 \sin x}{\sin x - 2 \cos x} dx$

Let  $\sin x - 2 \cos x = t$

$$\Rightarrow (\cos x + 2 \sin x) dx = dt$$

$$\therefore I_2 = 2 \int \frac{dt}{t} = 2 \ln t + C = 2 \ln(\sin x - 2 \cos x) + C$$

Hence,  $I_1 + I_2 = \int dx + 2 \ln(\sin x - 2 \cos x) + c$

$$= x + 2 \ln |(\sin x - 2 \cos x)| + k \Rightarrow a = 2$$

---

## Question287

If  $f(x) = \int \left( \frac{x^2 + \sin^2 x}{1 + x^2} \right) \sec^2 x dx$  and  $f(0) = 0$ , then  $f(1)$  equals

[Online May 19, 2012]

**Options:**

A.  $\tan 1 - \frac{\pi}{4}$

B.  $\tan 1 + 1$

C.  $\frac{\pi}{4}$

D.  $1 - \frac{\pi}{4}$

**Answer: A**

**Solution:**

**Solution:**

$$\text{Let } f(x) = \int \left( \frac{x^2 + \sin^2 x}{1 + x^2} \right) \sec^2 x dx$$

$$= \int \frac{x^2 \sec^2 x + \frac{\sin^2 x}{\cos^2 x}}{1 + x^2} dx$$

$$= \int \frac{x^2 \sec^2 x + \tan^2 x}{1 + x^2} dx$$

$$= \int \frac{x^2(1 + \tan^2 x) + \tan^2 x}{1 + x^2} dx$$

$$= \int \frac{x^2 + \tan^2 x(1 + x^2)}{1 + x^2} dx$$

$$= \int \frac{x^2}{1 + x^2} dx + \int \tan^2 x dx$$

$$= \int \frac{x^2 + 1 - 1}{1 + x^2} dx + \int (\sec^2 x - 1) dx$$

$$= \int 1 dx - \int \frac{dx}{1 + x^2} + \int \sec^2 x dx - \int dx$$

$$= -\tan^{-1} x + \tan x + c$$

Given :  $f(0) = 0$

$$\Rightarrow f(0) = -\tan^{-1} 0 + \tan 0 + c \Rightarrow c = 0$$

$$\therefore f(x) = -\tan^{-1} x + \tan x$$

$$\text{Now, } f(1) = -\tan^{-1}(1) + \tan 1 = \tan 1 - \frac{\pi}{4}$$

---

## Question288

The integral of  $\frac{x^2 - x}{x^3 - x^2 + x - 1}$  w.r.t.  $x$  is

[Online May 12, 2012]

Options:

A.  $\frac{1}{2} \log(x^2 + 1) + C$

B.  $\frac{1}{2} \log x^2 - 1 \mid + C$

C.  $\log(x^2 + 1) + C$

D.  $\log x^2 - 1 \mid + C$

**Answer: A**

**Solution:**

**Solution:**

$$\begin{aligned} \text{Let } I &= \int \frac{x^2 - x}{x^3 - x^2 + x - 1} dx \\ &= \int \frac{x(x-1)}{x^2(x-1) + (x-1)} dx = \int \frac{xdx}{x^2+1} = \frac{1}{2} \int \frac{2xdx}{(x^2+1)} \end{aligned}$$

$$\text{Let } x^2 + 1 = t \Rightarrow 2xdx = dt$$

$$\therefore I = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log t + c$$

$$= \frac{1}{2} \log(x^2 + 1) + c$$

where 'c' is the constant of integration.

---

## Question289

Let  $f(x)$  be an indefinite integral of  $\cos^3 x$ .

**Statement 1:**  $f(x)$  is a periodic function of period  $\pi$ .

**Statement 2:**  $\cos^3 x$  is a periodic function.

[Online May 7, 2012]

**Options:**

A. Statement 1 is true, Statement 2 is false.

B. Both the Statements are true, but Statement 2 is not the correct explanation of Statement 1.

C. Both the Statements are true, and Statement 2 is correct explanation of Statement 1.

D. Statement 1 is false, Statement 2 is true.

**Answer: D**

**Solution:**

**Solution:**

Statement -2 :  $\cos^3 x$  is a periodic function.

It is a true statement.

Statement -1

$$\text{Given } f(x) = \int \cos^3 x dx = \int \left( \frac{\cos 3x}{4} + \frac{3 \cos x}{4} \right) dx$$

$$= \frac{1}{4} \frac{\sin 3x}{3} + \frac{3}{4} \sin x = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x$$

$$\text{Now, period of } \frac{1}{12} \sin 3x = \frac{2\pi}{3}$$

$$\text{Period of } \frac{3}{4} \sin x = 2\pi$$

$$\text{Hence period of } f(x) = \frac{\text{L.C.M.}(2\pi, 2\pi)}{\text{HCF of}(1, 3)} = \frac{2\pi}{1} = 2\pi$$

Thus,  $f(x)$  is a periodic function of period  $2\pi$ .

Hence, Statement - 1 is false.

---

## Question290

If  $g(x) = \int_0^x \cos 4t dt$ , then  $g(x + \pi)$  equals

[2012]

**Options:**

A.  $\frac{g(x)}{g(\pi)}$

B.  $g(x) + g(\pi)$

C.  $g(x) - g(\pi)$

D.  $g(x) \cdot g(\pi)$

**Answer: 0**

**Solution:**

$$g(x + \pi) = \int_0^{x+\pi} \cos 4t \, dt$$

$$= \int_0^x \cos 4t \, dt + \int_0^{x+\pi} \cos 4t \, dt = g(x) + \int_0^{\pi} \cos 4t \, dt$$

(it is clear from graph of  $\cos 4t$ )

$$\int_0^{x+\pi} \cos 4t \, dt = \int_0^{\pi} \cos 4t \, dt = g(x) + g(\pi) = g(x) - g(\pi)$$

( $\because$  From graph of  $\cos 4t$ ,  $g(\pi) = 0$ )

---

## Question291

If  $[x]$  is the greatest integer  $\leq x$ , then the value of the integral

$$\int_{-0.9}^{0.9} \left( [x^2] + \log \left( \frac{2-x}{2+x} \right) \right) dx \text{ is}$$

[Online May 26, 2012]

Options:

- A. 0.486
- B. 0.243
- C. 1.8
- D. 0

Answer: D

Solution:

Solution:

$$\int_{-0.9}^{0.9} \left\{ [x^2] + \log \left( \frac{2-x}{2+x} \right) \right\} dx$$

$$= \int_{-0.9}^{0.9} [x^2] dx + \int_{-0.9}^{0.9} \log \left( \frac{2-x}{2+x} \right) dx$$

$$= 0 + \int_{-0.9}^{0.9} \log \left( \frac{2-x}{2+x} \right) dx$$

Put  $x = -x \Rightarrow f(x) = \log \frac{2-x}{2+x}$

and  $f(-x) = \log \frac{2+x}{2-x} = -\log \frac{(2-x)}{2+x} = -f(x)$

So, it is an odd function,  
hence Required integral = 0

---

## Question292

The value of the integral  $\int_0^{0.9} [x - 2[x]] dx$ , where  $[.]$  denotes the greatest integer function is  
[Online May 19, 2012]

Options:

- A. 0.9
- B. 1.8
- C. -0.9
- D. 0

**Answer: D**

**Solution:**

**Solution:**

Since  $\int_0^a [x] = 0$  where  $0 \leq a \leq 1$

$$\therefore \int_0^{0.9} [x - 2[x]] dx = 0$$

---

## Question293

If  $\frac{d}{dx} G(x) = \frac{e^{\tan x}}{x}$ ,  $x \in (0, \pi/2)$ , then  $\int_{\pi/4}^{\pi/2} \frac{2}{x} \cdot e^{\tan(\pi x^2)} dx$  is equal to  
[Online May 12, 2012]

Options:

- A.  $G(\pi/4) - G(\pi/16)$

B.  $2[G(\pi/4) - G(\pi/16)]$

C.  $\pi[G(1/2) - G(1/4)]$

D.  $G(1/\sqrt{2}) - G(1/2)$

**Answer: A**

**Solution:**

**Solution:**

$$\text{Let } \frac{d}{dx}G(x) = \frac{e^{\tan x}}{x}, x \in \left(0, \frac{\pi}{2}\right)$$

$$\text{Now, } I = \int_{1/4}^{1/2} \frac{2}{x} e^{\tan \pi x^2} \cdot dx = \int_{1/4}^{1/2} \frac{2\pi x}{\pi x^2} e^{\tan \pi x^2} dx$$

$$\text{Let } \pi x^2 = t \Rightarrow 2\pi x dx = dt$$

$$\text{When } x = \frac{1}{2}, t = \frac{\pi}{4} \text{ and } x = \frac{1}{4}, t = \frac{\pi}{16}$$

$$\therefore I = \int_{\pi/16}^{\pi/4} \frac{e^{\tan t}}{t} dt = G(t) \Big|_{\pi/16}^{\pi/4} = G\left(\frac{\pi}{4}\right) - G\left(\frac{\pi}{16}\right)$$

---

## Question294

If  $\int_e^x t f(t) dt = \sin x - x \cos x - \frac{x^2}{2}$ , for all  $x \in \mathbb{R} - \{0\}$ , then the value of  $f\left(\frac{\pi}{6}\right)$  is

[Online May 7, 2012]

**Options:**

A.  $1/2$

B.  $1$

C.  $0$

D.  $-1/2$

**Answer: D**



## Solution:

### Solution:

$$\text{Let } \int_e^x tf(t)dt = \sin x - x \cos x - \frac{x^2}{2}$$

By using Leibnitz rule, we get

$$\frac{d}{dx} \left[ \int_e^x tf(t)dt \right] = \frac{d}{dx} \left[ \sin x - x \cos x - \frac{x^2}{2} \right]$$

$$\Rightarrow xf(x) - ef(e).0 = x \sin x - x$$

Now, put  $x = \frac{\pi}{6}$ , we get

$$\frac{\pi}{6} \cdot f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} \cdot \sin \frac{\pi}{6} - \frac{\pi}{6}$$

$$\Rightarrow f\left(\frac{\pi}{6}\right) = \sin \frac{\pi}{6} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

---

## Question295

$f(x) = \int \frac{dx}{\sin^6 x}$  is a polynomial of degree

[Online May 26, 2012]

### Options:

A. 5 in cot x

B. 5 in tan x

C. 3 in tan x

D. 3 in cot x

**Answer: A**

### Solution:

#### Solution:

$$\text{Let } f(x) = \int \frac{dx}{\sin^6 x}$$

$$f(x) = \int \operatorname{cosec}^6 x dx$$

From reduction formula, we have

$$I_n = \int \operatorname{cosec}^n x dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\begin{aligned}
\therefore f(x) &= -\frac{\operatorname{cosec}^4 x \cot x}{5} + \frac{4}{5} \left[ \frac{-\operatorname{cosec}^2 x \cot x}{3} + \frac{2}{3} I_2 \right] \\
&= -\frac{\operatorname{cosec}^4 x \cot x}{5} - \frac{4}{15} \operatorname{cosec}^2 x \cdot \cot x + \frac{8}{15} [-\cot x] \\
&= \frac{-(1 + \cot^2 x)^2 \cdot \cot x}{5} - \frac{4}{15} (1 + \cot^2 x) \cot x - \frac{8}{15} (-\cot x) (\because \operatorname{cosec}^2 x = 1 + \cot^2 x) \\
&= \frac{-1}{5} [1 + \cot^4 x + 2\cot^2 x] \cot x - \frac{4}{15} [\cot x + \cot^3 x] - \frac{8}{15} \cot x \\
&= \frac{-1}{5} [\cot x + \cot^5 x + 2\cot^3 x] - \frac{4}{15} \cot x - \frac{4}{15} \cot^3 x - \frac{8}{15} \cot x \\
&= \frac{-15}{15} \cot x - \frac{\cot^5 x}{5} - \frac{10}{15} \cot^3 x \\
&= \frac{-\cot^5 x}{5} - \frac{2}{3} \cot^3 x - \cot x
\end{aligned}$$

It is a polynomial of degree 5 in  $\cot x$ .

---

## Question 296

Let  $[.]$  denote the greatest integer function then the value of  $\int_0^{1.5} x[x^2] dx$

is

[2011 RS]

Options:

A. 0

B.  $\frac{3}{2}$

C.  $\frac{3}{4}$

D.  $\frac{5}{4}$

**Answer: C**

**Solution:**

$$\int_0^{1.5} x[x^2]dx = \int_0^1 x[x^2]dx + \int_1^{\sqrt{2}} x[x^2]dx + \int_{\sqrt{2}}^{1.5} x[x^2]dx = \int_0^1 x \cdot 0dx + \int_1^{\sqrt{2}} xdx + \int_{\sqrt{2}}^{1.5} 2xdx = 0 + \left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + [x^2]_{\sqrt{2}}^{1.5}$$

$$= \frac{1}{2}(2-1) + (2.25-2) = \frac{1}{2} + 0.25$$

$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$


---

## Question 297

The value of  $\int_0^1 \frac{8 \log(1+x)}{1+x^2} dx$  is

[2011]

Options:

A.  $\frac{\pi}{8} \log 2$

B.  $\frac{\pi}{2} \log 2$

C.  $\log 2$

D.  $\pi \log 2$

Answer: D

Solution:

Solution:

$$\int_0^1 \frac{8 \log(1+x)}{1+x^2} dx$$

Put  $x = \tan \theta$

$$\therefore dx = \sec^2 \theta d\theta$$

$$\therefore I = 8 \int_0^{\pi/4} \frac{\log(1+\tan \theta)}{1+\tan^2 \theta} \cdot \sec^2 \theta d\theta$$

$$I = 8 \int_0^{\pi/4} \log(1+\tan \theta) d\theta \dots\dots(i)$$

Applying  $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$= 8 \int_0^{\pi/4} \log \left[ 1 + \tan \left( \frac{\pi}{4} - \theta \right) \right] d\theta$$

$$= 8 \int_0^{\pi/4} \log \left[ 1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right] d\theta = 8 \int_0^{\pi/4} \log \left[ \frac{2}{1 + \tan \theta} \right] d\theta$$

$$\begin{aligned}
&= 8 \int_0^{\pi/4} [\log 2 - \log(1 + \tan \theta)] d \theta \\
&= 8 \log 2 \int_0^{\pi/4} 1 d \theta - 8 \int_0^{\pi/4} \log(1 + \tan \theta) d \theta \\
I &= 8 \cdot (\log 2)[x]_0^{\pi/4} - 8 \int_0^{\pi/4} \log(1 + \tan \theta) d \theta \\
I &= 8 \cdot \frac{\pi}{4} \cdot \log 2 - I \text{ [From equation (i)]} \\
\Rightarrow 2I &= 2\pi \log 2 \\
\therefore I &= \pi \log 2
\end{aligned}$$


---

## Question298

Let  $p(x)$  be a function defined on  $\mathbb{R}$  such that  $p'(x) = p'(1 - x)$ , for all  $x \in [0, 1]$ ,  $p(0) = 1$  and  $p(1) = 41$  Then  $\int_0^1 p(x) dx$  equals [2010]

Options:

- A. 21
- B. 41
- C. 42
- D.  $\sqrt{41}$

**Answer: A**

**Solution:**

$$\begin{aligned}
p'(x) &= p'(1 - x) \\
\Rightarrow p(x) &= -p(1 - x) + c \\
\text{at } x &= 0 \\
p(0) &= -p(1) + c \Rightarrow 42 = c \\
\text{Now, } p(x) &= -p(1 - x) + 42 \\
\Rightarrow p(x) + p(1 - x) &= 42 \\
\text{Let } I &= \int_0^1 p(x) dx \dots\dots(i) \\
\Rightarrow I &= \int_0^1 p(1 - x) dx \dots\dots(ii)
\end{aligned}$$

Adding eqn. (i) and (ii),

$$2I = \int_0^1 (42) dx \Rightarrow I = 21$$

---

## Question299

$\int_0^{\pi} [\cot x] dx$ , where  $[.]$  denotes the greatest integer function, is equal to :  
[2009]

Options:

- A. 1
- B. -1
- C.  $-\frac{\pi}{2}$
- D.  $\frac{\pi}{2}$

Answer: C

Solution:

Solution:

$$\text{Let } I = \int_0^{\pi} [\cot x] dx \dots\dots(i)$$

$$= \int_0^{\pi} [\cot(\pi - x)] dx = \int_0^{\pi} [-\cot x] dx \dots\dots(ii)$$

Adding eq<sup>n</sup>s(i)& (ii),

We get

$$2I = \int_0^{\pi} ([\cot x] + [-\cot x]) dx$$

$$= \int_0^{\pi} (-1) dx$$

$$[\because [x] + [-x] = -1, \text{ if } x \notin \mathbb{Z} \text{ and } [x] + [-x] = 0, \text{ if } x \in \mathbb{Z}]$$

$$= [-x]_0^{\pi} = -\pi \Rightarrow I = -\frac{\pi}{2}$$

---

## Question300

The value of  $\sqrt{2} \int \frac{\sin x \, dx}{\sin\left(x - \frac{\pi}{4}\right)}$  is

[2008]

Options:

A.  $x + \log \left| \cos \left( x - \frac{\pi}{4} \right) \right| + c$

B.  $x - \log \left| \sin \left( x - \frac{\pi}{4} \right) \right| + c$

C.  $x + \log \left| \sin \left( x - \frac{\pi}{4} \right) \right| + c$

D.  $x - \log \left| \cos \left( x - \frac{\pi}{4} \right) \right| + c$

Answer: C

Solution:

Solution:

$$\text{Let } I = \sqrt{2} \int \frac{\sin x \, dx}{\sin\left(x - \frac{\pi}{4}\right)}$$

$$\text{Let } x - \frac{\pi}{4} = t \Rightarrow dx = dt$$

$$\Rightarrow I = \sqrt{2} \int \frac{\sin\left(t + \frac{\pi}{4}\right)}{\sin t} dt = \frac{\sqrt{2}}{\sqrt{2}} \int \left( \frac{\sin t + \cos t}{\sin t} \right) dt$$

$$\Rightarrow I = \int (1 + \cot t) dt = t + \log |\sin t| + c_1$$

$$= x - \frac{\pi}{4} + \log \left| \sin \left( x - \frac{\pi}{4} \right) \right| + c_1$$

$$= x + \log \left| \sin \left( x - \frac{\pi}{4} \right) \right| + c \quad \left( \text{where } c = c_1 - \frac{\pi}{4} \right)$$

# Question301

Let  $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx$  and  $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx$ . Then which one of the following is true?

[2008]

Options:

A.  $I > \frac{2}{3}$  and  $J > 2$

B.  $I < \frac{2}{3}$  and  $J < 2$

C.  $I < \frac{2}{3}$  and  $J > 2$

D.  $I > 23$  and  $J < 2$

**Answer: B**

**Solution:**

**Solution:**

We know that  $\frac{\sin x}{x} < 1$ , for  $x \in (0, 1)$

$$\Rightarrow \frac{\sin x}{\sqrt{x}} < \sqrt{x} \text{ on } x \in (0, 1)$$

$$\Rightarrow \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \sqrt{x} dx = \left[ \frac{2x^{3/2}}{3} \right]_0^1$$

$$\Rightarrow \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \frac{2}{3} \Rightarrow I < \frac{2}{3}$$

Also  $\frac{\cos x}{\sqrt{x}} < \frac{1}{\sqrt{x}}$  for  $x \in (0, 1)$

$$\Rightarrow \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 x^{-1/2} dx = [2\sqrt{x}]_0^1 = 2$$

$$\Rightarrow \int_0^1 \frac{\cos x}{\sqrt{x}} dx < 2 \Rightarrow J < 2$$

## Question302

$\int \frac{dx}{\cos x + \sqrt{3} \sin x}$  equals  
[2007]

Options:

A.  $\log \tan \left( \frac{x}{2} + \frac{\pi}{12} \right) + C$

B.  $\log \tan \left( \frac{x}{2} - \frac{\pi}{12} \right) + C$

C.  $\frac{1}{2} \log \tan \left( \frac{x}{2} + \frac{\pi}{12} \right) + C$

D.  $\frac{1}{2} \log \tan \left( \frac{x}{2} - \frac{\pi}{12} \right) + C$

Answer: C

Solution:

$$\begin{aligned} I &= \int \frac{dx}{\cos x + \sqrt{3} \sin x} \\ \Rightarrow I &= \int \frac{dx}{2 \left[ \frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x \right]} \\ &= \frac{1}{2} \int \frac{dx}{\left[ \sin \frac{\pi}{6} \cos x + \cos \frac{\pi}{6} \sin x \right]} = \frac{1}{2} \int \frac{dx}{\sin \left( x + \frac{\pi}{6} \right)} \\ \Rightarrow I &= \frac{1}{2} \int \operatorname{cosec} \left( x + \frac{\pi}{6} \right) dx \end{aligned}$$

We know that



$$\int \operatorname{cosec} x \, dx = \log |(\tan x/2)| + C$$

$$\therefore I = \frac{1}{2} \cdot \log \tan \left( \frac{x}{2} + \frac{\pi}{12} \right) + C$$

---

## Question303

The solution for x of the equation  $\int_{\sqrt{2}}^x \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2}$  is

[2007]

Options:

A.  $\frac{\sqrt{3}}{2}$

B.  $2\sqrt{2}$

C. 2

D. None of these

**Answer: D**

**Solution:**

**Solution:**

$$\int_{\sqrt{2}}^x \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2}$$

$$\therefore [\sec^{-1}t]_{\sqrt{2}}^x = \pi/2 \quad \left[ \because \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x \right]$$

$$\Rightarrow \sec^{-1}x - \sec^{-1}\sqrt{2} = \frac{\pi}{2}$$

$$\Rightarrow \sec^{-1}x - \frac{\pi}{4} = \frac{\pi}{2} \Rightarrow \sec^{-1}x = \frac{\pi}{2} + \frac{\pi}{4}$$

$$\Rightarrow \sec^{-1}x = \frac{3\pi}{4} \Rightarrow x = \sec \frac{3\pi}{4} = \sec \left( \pi - \frac{\pi}{4} \right)$$

$$\Rightarrow x = -\sec \frac{\pi}{4} \Rightarrow x = -\sqrt{2}$$

---

## Question304

Let  $F(x) = f(x) + f\left(\frac{1}{x}\right)$ , where  $f(x) = \int_1^x \frac{\log t}{1+t} dt$ , Then  $F(e)$  equals

[2007]

Options:

- A. 1
- B. 2
- C. 1/2
- D. 0

Answer: C

Solution:

Solution:

Given that  $F(x) = f(x) + f\left(\frac{1}{x}\right)$ , where

$$f(x) = \int_1^x \frac{\log t}{1+t} dt$$

$$\therefore F(e) = f(e) + f\left(\frac{1}{e}\right)$$

$$\Rightarrow F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{1+t} dt \dots\dots(1)$$

$$\text{Let } I = \int_1^{1/e} \frac{\log t}{1+t} dt$$

$$\therefore \text{Put } \frac{1}{t} = z \Rightarrow -\frac{1}{t^2} dt = dz \Rightarrow dt = -\frac{dz}{z^2}$$

$$\text{when } t = 1 \Rightarrow z = 1 \text{ and when } t = \frac{1}{e}$$

$$\Rightarrow z = e$$

$$\therefore I = \int_1^e \frac{\log\left(\frac{1}{z}\right)}{1+\frac{1}{z}} \left(-\frac{dz}{z^2}\right)$$

$$= \int_1^e \frac{(\log 1 - \log z) \cdot z}{z+1} \left(-\frac{dz}{z^2}\right)$$

$$= \int_1^e -\frac{\log z}{(z+1)} \left(-\frac{dz}{z}\right) [\because \log 1 = 0]$$

$$= \int_1^e \frac{\log z}{z(z+1)} dz$$

$$\therefore I = \int_1^e \frac{\log t}{t(t+1)} dt$$



[By property  $\int_a^b f(t) dt = \int_a^b f(x) dx$  ]

Now from eqn. (1)

$$F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^e \frac{\log t}{t(1+t)} dt$$

$$= \int_1^e \frac{t \cdot \log t + \log t}{t(1+t)} dt = \int_1^e \frac{(\log t)(t+1)}{t(1+t)} dt$$

$$\Rightarrow F(e) = \int_1^e \frac{\log t}{t} dt$$

Let  $\log t = x \therefore \frac{1}{t} dt = dx$

[when  $t = 1, x = 0$  and when  $t = e, x = \log e = 1$  ]

$$\therefore F(e) = \int_0^1 x dx = \left[ \frac{x^2}{2} \right]_0^1$$

$$\Rightarrow F(e) = \frac{1}{2}$$

## Question305

The value of  $\int_1^a [x]f'(x) dx$ ,  $a > 1$  where  $[x]$  denotes the greatest integer not exceeding  $x$  is  
[2006]

Options:

- A.  $af(a) - \{f(1) + f(2) + \dots + f([a])\}$
- B.  $[a]f(a) - \{f(1) + f(2) + \dots + f([a])\}$
- C.  $[a]f([a]) - \{f(1) + f(2) + \dots + f(a)\}$
- D.  $af([a]) - \{f(1) + f(2) + \dots + f(a)\}$

**Answer: B**

**Solution:**

**Solution:**

Let  $a = k + h$  where  $k$  is an integer such that and  $0 \leq h < 1$

$$\Rightarrow [a] = k$$

$$\therefore \int_1^a [x]f'(x) dx = \int_1^2 1f'(x) dx + \int_2^3 2f'(x) dx + \dots + \int_{k-1}^k (k-1)f'(x) dx + \int_k^{k+h} kf'(x) dx$$



$$\begin{aligned}
&= \{f(2) - f(1)\} + 2\{f(3) - f(2)\} + 3\{f(4) - f(3)\} + \dots + (k-1)\{f(k) - f(k-1)\} + k\{f(k+h) - f(k)\} \\
&= -f(1) - f(2) - f(3) \dots - f(k) + kf(k+h) \\
&= [a]f(a) - \{f(1) + f(2) + f(3) + \dots + f([a])\}
\end{aligned}$$


---

## Question306

$$\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x + \pi)^3 + \cos^2(x + 3\pi)] dx \text{ is equal to}$$

[2006]

Options:

A.  $\frac{\pi^4}{32}$

B.  $\frac{\pi^4}{32} + \frac{\pi}{2}$

C.  $\frac{\pi}{2}$

D.  $\frac{\pi}{4} - 1$

Answer: C

Solution:

Solution:

$$\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x + \pi)^3 + \cos^2(x + 3\pi)] dx$$

Put  $x + \pi = t$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [t^3 + \cos^2 t] dt = 2 \int_0^{\frac{\pi}{2}} \cos^2 t dt$$

[  $\because t^3$  is odd and  $\cos^2 t$  is even function ]

$$= \int_0^{\frac{\pi}{2}} (1 + \cos 2t) dt = \frac{\pi}{2} + 0$$

---

## Question307

$\int_0^{\pi} x f(\sin x) dx$  is equal to  
[2006]

Options:

A.  $\pi \int_0^{\pi} f(\cos x) dx$

B.  $\pi \int_0^{\pi} f(\sin x) dx$

C.  $\frac{\pi}{2} \int_0^{\pi/2} f(\sin x) dx$

D.  $\pi \int_0^{\pi/2} f(\cos x) dx$

**Answer: D**

**Solution:**

**Solution:**

$$I = \int_0^{\pi} x f(\sin x) dx = \int_0^{\pi} (\pi - x) f(\sin x) dx$$

$$= \pi \int_0^{\pi} f(\sin x) dx - I \Rightarrow 2I = \pi \int_0^{\pi} f(\sin x) dx$$

$$I = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx = \pi \int_0^{\pi/2} f(\sin x) dx [\because \sin(\pi - x) = \sin x]$$

$$= \pi \int_0^{\pi/2} f(\cos x) dx$$

---

## Question308

The value of integral,  $\int_3^6 \frac{\sqrt{x}}{\sqrt{9-x} + \sqrt{x}} dx$  is

[2006]

Options:

A.  $\frac{1}{2}$

B.  $\frac{3}{2}$

C. 2

D. 1

Answer: B

Solution:

Solution:

$$I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x} + \sqrt{x}} dx \dots (1)$$

$$I = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{x}} dx \dots (2)$$

$$\left[ \because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

Adding equation (1) and (2)

$$2I = \int_3^6 dx = [x]_3^6 = 3 \Rightarrow I = \frac{3}{2}$$

---

## Question309

$\int \left\{ \frac{(\log x - 1)}{1 + (\log x)^2} \right\}^2 dx$  is equal to

[2005]

Options:

A.  $\frac{\log x}{(\log x)^2 + 1} + C$



B.  $\frac{x}{x^2+1} + C$

C.  $\frac{xe^x}{1+x^2} + C$

D.  $\frac{x}{(\log x)^2+1} + C$

**Answer: D**

**Solution:**

**Solution:**

$$\int \frac{(\log x - 1)^2}{(1 + (\log x)^2)^2} dx = \int \frac{1 + (\log x)^2 - 2 \log x}{[1 + (\log x)^2]^2} dx$$

$$= \int \left[ \frac{1}{(1 + (\log x)^2)} - \frac{2 \log x}{(1 + (\log x)^2)^2} \right] dx$$

$$\therefore I = \int \left[ \frac{e^t}{1+t^2} - \frac{2te^t}{(1+t^2)^2} \right] dt$$

$$= \int e^t \left[ \frac{1}{1+t^2} - \frac{2t}{(1+t^2)^2} \right] dt$$

[ Which is of the form  $\int e^x(f(x) + f'(x))dx = f(x) \cdot e^x + c$  ]

$$= \frac{e^t}{1+t^2} + c = \frac{x}{1 + (\log x)^2} + c$$

## Question310

The value of  $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$ ,  $a > 0$ , is

[2005]

**Options:**

A.  $a\pi$

B.  $\frac{\pi}{2}$

C.  $\frac{\pi}{a}$

D.  $2\pi$

**Answer: B**

**Solution:**

**Solution:**

$$\text{Let } I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx \dots\dots\dots(1)$$

$$I = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^x} dx \text{ Using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \\ = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx \dots\dots\dots(2)$$

Adding equations (1) and (2) we get

$$2I = \int_{-\pi}^{\pi} \cos^2 x \left( \frac{1+a^x}{1+a^x} \right) dx = \int_{-\pi}^{\pi} \cos^2 x dx \\ = 2 \int_0^{\pi} \cos^2 x dx [\because f(\pi-x) = f(x)] \\ = 2 \times 2 \int_0^{\frac{\pi}{2}} \cos^2 x dx = 4 \int_0^{\frac{\pi}{2}} \sin^2 x dx \left[ \because f\left(\frac{\pi}{2}-x\right) = f(x) \right] \\ \Rightarrow I = 2 \int_0^{\frac{\pi}{2}} \sin^2 x dx = 2 \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) dx \\ \Rightarrow I = 2 \int_0^{\frac{\pi}{2}} dx - 2 \int_0^{\frac{\pi}{2}} \cos^2 x dx \\ \Rightarrow I + I = 2 \left( \frac{\pi}{2} \right) = \pi \Rightarrow I = \frac{\pi}{2}$$

---

## Question311

If  $I_1 = \int_0^1 2^{x^2} dx$ ,  $I_2 = \int_0^1 2^{x^3} dx$ ,  $I_3 = \int_1^2 2^{x^2} dx$  and  $I_4 = \int_1^2 2^{x^3} dx$  then  
[2005]

**Options:**

A.  $I_2 > I_1$

B.  $I_1 > I_2$



C.  $I_3 = I_4$

D.  $I_3 > I_4$

**Answer: B**

**Solution:**

**Solution:**

$$I_1 = \int_0^1 2^{x^2} dx, I_2 = \int_0^1 2^{x^3} dx, I_3$$

$$= \int_1^2 2^{x^2} dx, I_4 = \int_1^2 2^{x^3} dx$$

$$\because 2^{x^3} < 2^{x^2}, 0 < x < 1$$

$$\Rightarrow \int_0^1 2^{x^2} dx > \int_0^1 2^{x^3} dx \Rightarrow I_1 > I_2$$

$$\text{and } 2^{x^3} > 2^x, x > 1$$

$$\Rightarrow I_4 > I_3$$

---

## Question312

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function having  $f(2) = 6$ ,

$f'(2) = \left(\frac{1}{48}\right)$ . Then  $\lim_{x \rightarrow 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt$  equals

[2005]

**Options:**

A. 24

B. 36

C. 12

D. 18

**Answer: D**

**Solution:**

$$\lim_{x \rightarrow 2} \int_0^{f(x)} \frac{4t^3}{x-2} dt = \lim_{x \rightarrow 2} \frac{\int_0^{f(x)} 4t^3 dt}{x-2}$$

Applying L Hospital rule

$$\lim_{x \rightarrow 2} \frac{[4f(x)^3 f'(x)]}{1} = 4(f(2))^3 f'(2)$$

$$= 4 \times 6^3 \times \frac{1}{48} = 18$$

## Question313

$$\lim_{n \rightarrow \infty} \left[ \frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{4}{n^2} + \dots + \frac{1}{n} \sec^2 1 \right]$$

**equals**  
**[2005]**

**Options:**

- A.  $\frac{1}{2} \sec 1$
- B.  $\frac{1}{2} \operatorname{cosec} 1$
- C.  $\tan 1$
- D.  $\frac{1}{2} \tan 1$

**Answer: D**

**Solution:**

**Solution:**

$$\lim_{n \rightarrow \infty} \left[ \frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{4}{n^2} + \frac{3}{n^2} \sec^2 \frac{9}{n^2} + \dots + \frac{1}{n} \sec^2 1 \right] \text{ is equal to}$$

$$\lim_{n \rightarrow \infty} \frac{r}{n^2} \sec^2 \frac{r^2}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{r}{n} \sec^2 \frac{r^2}{n^2}$$

$$\Rightarrow \text{Given limit is equal to value of integral } \int_0^1 x \sec^2 x^2 dx$$

$$\text{or } \frac{1}{2} \int_0^1 2x \sec x^2 dx = \frac{1}{2} \int_0^1 \sec^2 t dt \text{ [put } x^2 = t \text{]} \\ = \frac{1}{2} (\tan t)_0^1 = \frac{1}{2} \tan 1$$


---

## Question 314

$\int \frac{dx}{\cos x - \sin x}$  is equal to  
[2004]

Options:

A.  $\frac{1}{\sqrt{2}} \log \left| \tan \left( \frac{x}{2} + \frac{3\pi}{8} \right) \right| + C$

B.  $\frac{1}{\sqrt{2}} \log \left| \cot \left( \frac{x}{2} \right) \right| + C$

C.  $\frac{1}{\sqrt{2}} \log \left| \tan \left( \frac{x}{2} - \frac{3\pi}{8} \right) \right| + C$

D.  $\frac{1}{\sqrt{2}} \log \left| \tan \left( \frac{x}{2} - \frac{\pi}{8} \right) \right| + C$

**Answer: A**

**Solution:**

**Solution:**

$$\int \frac{dx}{\cos x - \sin x} = \int \frac{dx}{\sqrt{2} \left( \frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x \right)} \\ = \int \frac{dx}{\sqrt{2} \cos \left( x + \frac{\pi}{4} \right)} = \frac{1}{\sqrt{2}} \int \sec \left( x + \frac{\pi}{4} \right) dx \\ = \frac{1}{\sqrt{2}} \log \left| \tan \left( \frac{\pi}{4} + \frac{x}{2} + \frac{\pi}{8} \right) \right| + C \left[ \because \int \sec x dx = \log \left| \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) \right| \right] \\ = \frac{1}{\sqrt{2}} \log \left| \tan \left( \frac{x}{2} + \frac{3\pi}{8} \right) \right| + C$$


---

## Question315

If  $\int \frac{\sin x}{\sin(x-\alpha)} dx = Ax + B \log \sin(x-\alpha) + C$ , then value of (A, B) is  
[2004]

Options:

A.  $(-\cos \alpha, \sin \alpha)$

B.  $(\cos \alpha, \sin \alpha)$

C.  $(-\sin \alpha, \cos \alpha)$

D.  $(\sin \alpha, \cos \alpha)$

Answer: B

Solution:

Solution:

$$\int \frac{\sin x}{\sin(x-\alpha)} dx = \int \frac{\sin(x-\alpha+\alpha)}{\sin(x-\alpha)} dx$$
$$= \int \frac{\sin(x-\alpha)\cos\alpha + \cos(x-\alpha)\sin\alpha}{\sin(x-\alpha)} dx$$

$$= \int \{\cos\alpha + \sin\alpha \cot(x-\alpha)\} dx$$

$$= (\cos\alpha)x + (\sin\alpha) \log \sin(x-\alpha) + C$$

Comparing with  $Ax + B \log \sin(x-\alpha) + c$

$$\therefore A = \cos \alpha, B = \sin \alpha$$

---

## Question316

If  $f(x) = \frac{e^x}{1+e^x}$ ,  $I_1 = \int_{f(-a)}^{f(a)} xg\{x(1-x)\} dx$  and  $I_2 = \int_{f(-a)}^{f(a)} g\{x(1-x)\} dx$ , then  
the value of  $\frac{I_2}{I_1}$  is

[2004]

Options:

A. 1

B. -3

C. -1

D. 2

**Answer: D**

**Solution:**

**Solution:**

$$f(x) = \frac{e^x}{1+e^x} \Rightarrow f(-x) = \frac{e^{-x}}{1+e^{-x}} = \frac{1}{e^x+1}$$

$$\therefore f(x) + f(-x) = 1 \quad \forall x \in \mathbb{R}$$

$$\text{Now } I_1 = \int_{f(-a)}^{f(a)} xg\{x(1-x)\} dx$$

$$= \int_{f(-a)}^{f(a)} (1-x)g\{x(1-x)\} dx \left[ \text{using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$\Rightarrow \int_{f(-a)}^{f(a)} g\{x(1-x)\} dx - \int_{f(-a)}^{f(a)} xg\{x(1-x)\} dx$$

$$= I_2 - I_1 \Rightarrow 2I_1 = I_2$$

---

## Question 317

If  $\int_0^\pi x f(\sin x) dx = A \int_0^{\pi/2} f(\sin x) dx$ , then A is  
[2004]

**Options:**

A.  $2\pi$

B.  $\pi$

C.  $\frac{\pi}{4}$

D. 0

**Answer: B**

**Solution:**

**Solution:**

$$\text{Let } I = \int_0^{\pi} x f(\sin x) dx \dots\dots(i)$$

We know that

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx = \int_0^{\pi} (\pi-x) f(\sin x) dx \dots\dots(ii)$$

Adding (i) and (ii)

$$\therefore 2I = \pi \int_0^{\pi} f(\sin x) dx = \pi \cdot 2 \int_0^{\pi/2} f(\sin x) dx [\because \sin(\pi-x) = \sin x]$$

$$\therefore I = \pi \int_0^{\pi/2} f(\sin x) dx \Rightarrow A = \pi$$

$$\text{Let } \log x = t \Rightarrow e^t = x$$

$$\Rightarrow \frac{1}{x} dx = dt \Rightarrow dx = x dt \Rightarrow e^t dt$$

## Question318

The value of  $I = \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{1 + \sin 2x}} dx$  is

[2004]

**Options:**

A. 3

B. 1

C. 2

D. 0

**Answer: C**

**Solution:**

**Solution:**

$$I = \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{1 + \sin 2x}} dx$$

$$\int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{(\sin x + \cos x)^2}}$$

$$I = \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{(\sin x + \cos x)} dx = \int_0^{\pi/2} (\sin x + \cos x) dx \quad [\because \sin x + \cos x > 0 \text{ if } 0 < x < \frac{\pi}{2}]$$

$$\text{or } I = [-\cos x + \sin x]_0^{\pi/2} = 2$$


---

## Question319

The value of  $\int_{-2}^3 |1 - x^2| dx$  is  
[2004]

Options:

A.  $\frac{1}{3}$

B.  $\frac{14}{3}$

C.  $\frac{7}{3}$

D.  $\frac{28}{3}$

**Answer: D**

**Solution:**

**Solution:**

$$\int_{-2}^3 |1 - x^2| dx = \int_{-2}^3 |x^2 - 1| dx$$

$$\text{Now } |x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } x \leq -1 \\ 1 - x^2 & \text{if } -1 \leq x \leq 1 \\ x^2 - 1 & \text{if } x \geq 1 \end{cases}$$

$$\begin{aligned} \therefore \text{Integral is } & \int_{-2}^{-1} (x^2 - 1) dx + \int_{-1}^1 (1 - x^2) dx + \int_1^3 (x^2 - 1) dx \\ & = \left[ \frac{x^3}{3} - x \right]_{-2}^{-1} + \left[ x - \frac{x^3}{3} \right]_{-1}^1 + \left[ \frac{x^3}{3} - x \right]_1^3 \\ & = \left( -\frac{1}{3} + 1 \right) - \left( -\frac{8}{3} + 2 \right) + \left( 2 - \frac{2}{3} \right) + \left( \frac{27}{3} - 3 \right) - \left( \frac{1}{3} - 1 \right) \\ & = \frac{2}{3} + \frac{2}{3} + \frac{4}{3} + 6 + \frac{2}{3} = \frac{28}{3} \end{aligned}$$

## Question320

$$\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} e^{\frac{r}{n}} \text{ is}$$

[2004]

**Options:**

A.  $e + 1$

B.  $e - 1$

C.  $1 - e$

D.  $e$

**Answer: B**

**Solution:**

**Solution:**

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} e^{\frac{r}{n}} & \text{ [Using definite integrals as limit of sum]} \\ & = \int_0^1 e^x dx = e - 1 \end{aligned}$$

---

## Question321

The value of the integral  $I = \int_0^1 x(1-x)^n dx$  is

[2003]

**Options:**

A.  $\frac{1}{n+1} + \frac{1}{n+2}$

B.  $\frac{1}{n+1}$



C.  $\frac{1}{n+2}$

D.  $\frac{1}{n+1} - \frac{1}{n+2}$ .

**Answer: D**

**Solution:**

**Solution:**

$$\begin{aligned} I &= \int_0^1 x(1-x)^n dx = \int_0^1 (1-x)(1-x)^n dx \\ &= \int_0^1 (1-x)x^n dx = \int_0^1 (x^n - x^{n+1}) dx \\ &= \left[ \frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right]_0^1 = \frac{1}{n+1} - \frac{1}{n+2} \end{aligned}$$

---

## Question322

Let  $f(x)$  be a function satisfying  $f'(x) = f(x)$  with  $f(0) = 1$  and  $g(x)$  be a function that satisfies  $f(x) + g(x) = x^2$ . Then the value of the integral

$\int_0^1 f(x)g(x) dx$ , is

[2003]

**Options:**

A.  $e + \frac{e^2}{2} + \frac{5}{2}$

B.  $e - \frac{e^2}{2} - \frac{5}{2}$

C.  $e + \frac{e^2}{2} - \frac{3}{2}$

D.  $e - \frac{e^2}{2} - \frac{3}{2}$ .

**Answer: D**

**Solution:**

**Solution:**

$$\text{Given that } f'(x) = f(x) \Rightarrow \frac{f'(x)}{f(x)} = 1$$

Integrating both side we get

$$\log f(x) = x + c \Rightarrow f(x) = e^{x+c}$$

$$f(0) = 1 \Rightarrow f(x) = e^x$$

$$\therefore g(x) = x^2 - f(x) = x^2 - e^x$$

$$\therefore \int_0^1 f(x)g(x)dx = \int_0^1 e^x(x^2 - e^x)dx$$

$$= \int_0^1 x^2 e^x dx - \int_0^1 e^{2x} dx$$

$$= [x^2 e^x]_0^1 - 2[xe^x - e^x]_0^1 - \frac{1}{2}[e^{2x}]_0^1$$

$$= e - \left[ \frac{e^2}{2} - \frac{1}{2} \right] - 2[e - e + 1] = e - \frac{e^2}{2} - \frac{3}{2}$$

---

## Question323

If  $f(a + b - x) = f(x)$  then  $\int_a^b xf(x)dx$  is equal to  
[2003]

**Options:**

A.  $\frac{a+b}{2} \int_a^b f(a+b+x)dx$

B.  $\frac{a+b}{2} \int_a^b f(b-x)dx$

C.  $\frac{a+b}{2} \int_a^b f(x)dx$

D.  $\frac{b-a}{2} \int_a^b f(x)dx$ .

**Answer: C**

**Solution:**

**Solution:**

$$I = \int_a^b xf(x)dx = \int_a^b (a+b-x)f(a+b-x)dx$$

We know that

$$\begin{aligned}
&\text{because } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \\
&= (a+b) \int_a^b f(a+b-x) dx - \int_a^b xf(a+b-x) dx \\
&= (a+b) \int_a^b f(x) dx - \int_a^b xf(x) dx \quad [\because \text{Given that } f(a+b-x) = f(x)] \\
2I &= (a+b) \int_a^b f(x) dx \\
\Rightarrow I &= \frac{(a+b)}{2} \int_a^b f(x) dx
\end{aligned}$$


---

## Question324

The value of  $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sec^2 t dt}{x \sin x}$  is

[2003]

Options:

- A. 0
- B. 3
- C. 2
- D. 1

**Answer: D**

**Solution:**

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\frac{d}{dx} \int_0^{x^2} \sec^2 t dt}{\frac{d}{dx} (x \sin x)} &= \lim_{x \rightarrow 0} \frac{\sec^2 x^2 \cdot 2x}{\sin x + x \cos x} \quad (\text{by L' Hospital rule}) \\
\lim_{x \rightarrow 0} \frac{2 \sec^2 x^2}{\left(\frac{\sin x}{x} + \cos x\right)} &= \frac{2 \times 1}{1 + 1} = 1
\end{aligned}$$


---

## Question325

If  $f(y) = e^y$ ,  $g(y) = y$ ;  $y > 0$  and  $F(t) = \int_0^t f(t-y)g(y)dy$ , then  
[2003]

Options:

- A.  $F(t) = te^{-t}$
- B.  $F(t) = 1 - te^{-t}(1+t)$
- C.  $F(t) = e^t - (1+t)$
- D.  $F(t) = te^t$

Answer: C

Solution:

Solution:

$$\begin{aligned} F(t) &= \int_0^t f(t-y)g(y)dy \\ &= \int_0^t e^{t-y}ydy = e^t \int_0^t e^{-y}ydy \\ &= e^t[-ye^{-y} - e^{-y}]_0^t = -e^t[ye^{-y} + e^{-y}]_0^t \\ &= -e^t[te^{-t} + e^{-t} - 0 - 1] = -e^t \left[ \frac{t+1-e^t}{e^t} \right] \\ &= e^t - (1+t) \end{aligned}$$

---

## Question326

$$\lim_{n \rightarrow \infty} \frac{1+2^4+3^4+\dots+n^4}{n^5} - \lim_{n \rightarrow \infty} \frac{1+2^3+3^3+\dots+n^3}{n^5}$$

[2003]

Options:

- A.  $\frac{1}{5}$

B.  $\frac{1}{30}$

C. Zero

D.  $\frac{1}{4}$

**Answer: A**

**Solution:**

**Solution:**

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1 + 2^4 + 3^4 + \dots + n^4}{n^5} - \lim_{n \rightarrow \infty} \frac{1 + 2^3 + 3^3 + \dots + n^3}{n^5} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n}\right)^4 - \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{r}{n}\right)^3 \\ &= \int_0^1 x^4 dx - \lim_{n \rightarrow \infty} \frac{1}{n} \times \int_0^1 x^3 dx = \left[\frac{x^5}{5}\right]_0^1 - 0 = \frac{1}{5} \end{aligned}$$

---

## Question327

**f(x) and g(x) are two differentiable functions on [0,2] such that  $f''(x) - g''(x) = 0$ ,  $f'(1) = 2g'(1) = 4f(2) = 3g(2) = 9$  then  $f(x) - g(x)$  at  $x = 3/2$  is**

**[2002]**

**Options:**

A. 0

B. 2

C. 10

D. 5

**Answer: D**

**Solution:**



$$\begin{aligned} \because f''(x) - g''(x) &= 0 \\ \text{Integrating, } f'(x) - g'(x) &= c; \\ \Rightarrow f'(1) - g'(1) = c &\Rightarrow 4 - 2 = c \Rightarrow c = 2 \\ \therefore f'(x) - g'(x) &= 2 \\ \text{Integrating, } f(x) - g(x) &= 2x + c_1 \\ \Rightarrow f(2) - g(2) = 4 + c_1 &\Rightarrow 9 - 3 = 4 + c_1 \\ \Rightarrow c_1 = 2 \therefore f(x) - g(x) &= 2x + 2 \\ \text{At } x = 3/2, f(x) - g(x) &= 3 + 2 = 5. \end{aligned}$$


---

## Question328

$$\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx \text{ is}$$

[2002]

**Options:**

A.  $\frac{\pi^2}{4}$

B.  $\pi^2$

C. zero

D.  $\frac{\pi}{2}$

**Answer: B**

**Solution:**

$$\begin{aligned} &\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx \\ &= \int_{-\pi}^{\pi} \frac{2xdx}{1 + \cos^2 x} + 2 \int_{-\pi}^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \\ &= 0 + 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \end{aligned}$$

We know that

$$\because \int_{-a}^a f(x) dx = 0, \text{ if } f(x) \text{ is odd.}$$

$$= 2 \int_0^a f(x) dx, \text{ if } f(x) \text{ is even}$$

$$I = 4 \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$

$$I = 4 \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx$$

$$\Rightarrow I = 4\pi \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x} - 4 \int_0^{\pi} \frac{x \sin x dx}{1 + \cos^2 x}$$

$$\Rightarrow 2I = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$

$$\text{put } \cos x = t \Rightarrow -\sin x dx = dt$$

$$\text{when } x = 0, t = 1 \text{ and when } x = \pi, t = -1$$

$$\therefore I = -2\pi \int_1^{-1} \frac{1}{1+t^2} dt = 2\pi \int_{-1}^1 \frac{1}{1+t^2} dt$$

$$= 2\pi [\tan^{-1} t]_{-1}^1 = 2\pi [\tan^{-1} 1 - \tan^{-1}(-1)]$$

$$= 2\pi \left[ \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right] = 2\pi \cdot \frac{\pi}{2} = \pi^2$$

## Question 329

$$\int_0^2 [x^2] dx \text{ is}$$

[2002]

**Options:**

A.  $2 - \sqrt{2}$

B.  $2 + \sqrt{2}$

C.  $\sqrt{2} - 1$

D.  $-\sqrt{2} - \sqrt{3} + 5$

**Answer: D**

**Solution:**

**Solution:**

We know that  $[x]$  is greatest integer function less than equal to  $x$

$$\therefore \int_0^2 [x^2] dx = \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2] dx + \int_{\sqrt{3}}^2 [x^2] dx$$

$$\begin{aligned}
&= \int_0^1 0 dx + \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 dx + \int_{\sqrt{3}}^2 3 dx \\
&= [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{\sqrt{3}} + [3x]_{\sqrt{3}}^2 \\
&= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3} \\
&= 5 - \sqrt{3} - \sqrt{2}
\end{aligned}$$


---

## Question 330

$I_n = \int_0^{\pi/4} \tan^n x dx$  then  $\lim_{n \rightarrow \infty} n[I_n + I_{n+2}]$  equals  
[2002]

Options:

A.  $\frac{1}{2}$

B. 1

C. of

D. zero

**Answer: B**

**Solution:**

**Solution:**

$$\begin{aligned}
I_n + I_{n+2} &= \int_0^{\pi/4} \tan^n x (1 + \tan^2 x) dx \\
&= \int_0^{\pi/4} \tan^n x \sec^2 x dx = \left[ \frac{\tan^{n+1} x}{n+1} \right]_0^{\pi/4} \left[ \because \int x^n dx = \frac{x^{n+1}}{n+1} \right] \\
&= \frac{1-0}{n+1} = \frac{1}{n+1} \\
\therefore I_n + I_{n+2} &= \frac{1}{n+1} \Rightarrow \lim_{n \rightarrow \infty} n[I_n + I_{n+2}] \\
&= \lim_{n \rightarrow \infty} n \cdot \frac{1}{n+1} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{n}{n \left( 1 + \frac{1}{n} \right)} = 1
\end{aligned}$$


---



## Question331

$\int_0^{10\pi} |\sin x| dx$  is  
[2002]

**Options:**

A. 20

B. 8

C. 10

D. 18

**Answer: A**

**Solution:**

**Solution:**

$$I = \int_0^{10\pi} |\sin x| dx = 10 \int_0^{\pi} |\sin x| dx \quad [\because \sin(10\pi - x) = \sin x]$$
$$= \int_0^{10\pi} |\sin x| dx$$

$\because \sin x > 0$ , for  $0 < x < \pi$

as  $\sin(\pi - x) = \sin x$

$$I = 20 \int_0^{\pi/2} \sin x dx = 20[-\cos x]_0^{\pi/2} = 20$$

---

## Question332

$\lim_{n \rightarrow \infty} \frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}}$  is  
[2002]

**Options:**

A.  $\frac{1}{p+1}$

B.  $\frac{1}{1-p}$

$$C. \frac{1}{p} - \frac{1}{p-1}$$

$$D. \frac{1}{p+2}$$

**Answer: A**

**Solution:**

**Solution:**

We have  $\lim_{n \rightarrow \infty} \frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}};$

$$\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r^p}{n^p \cdot n} = \int_0^1 x^p dx = \left[ \frac{x^{p+1}}{p+1} \right]_0^1 = \frac{1}{p+1}$$

---